Conjectures equivalent to the Borodin-Kostochka conjecture that a priori seem weaker

Daniel W. Cranston* Landon Rabern†

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Abstract

Borodin and Kostochka conjectured that every graph G with maximum degree $\Delta \geq 9$ satisfies $\chi \leq \max\{\omega, \Delta - 1\}$. We carry out an in-depth study of minimum counterexamples to the Borodin-Kostochka conjecture. Our main tool is the classification of graph joins $A * B$ with $|A| \geq 2$, $|B| \geq 2$ which are f-choosable, where $f(v) := d(v) - 1$ for each vertex v. Since such a join cannot be an induced subgraph of a vertex critical graph with $\chi = \Delta$, we have a wealth of structural information about minimum counterexamples to the Borodin-Kostochka conjecture.

Our main result is to prove that certain conjectures that are a priori weaker than the Borodin-Kostochka Conjecture are in fact equivalent to it. One such equivalent conjecture is the following: Any graph with $\chi \geq \Delta = 9$ contains $K_3 * E_6$ as a subgraph.

1 Introduction

1.1 A short history of the problem

The first non-trivial result about coloring graphs with around Δ colors is Brooks' theorem from 1941.

Theorem 1.1 (Brooks [4]). *Every graph with* $\Delta \geq 3$ *satisfies* $\chi \leq \max\{\omega, \Delta\}$.

In 1977, Borodin and Kostochka conjectured that a similar result holds for $\Delta - 1$ colorings. Counterexamples exist showing that the $\Delta \geq 9$ condition is tight (see Figures 1, 2, 3 and 4).

Conjecture 1.2 (Borodin and Kostochka [3]). *Every graph with* $\Delta \geq 9$ *satisfies* $\chi \leq \max\{\omega, \Delta - 1\}$.

In the same paper they proved the following weakening. The proof is quite simple once you have a decomposition lemma of Lovász from the 1960's [12].

*Department of Mathematics and Applied Mathematics, Virginia Commonwealth University, Richmond, VA, 23284. email: dcranston@vcu.edu

†School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287. email: landon.rabern@gmail.com.

Theorem 1.3 (Borodin and Kostochka [3]). *Every graph satisfying $\chi \geq \Delta \geq 7$ contains a $K_{\lfloor \frac{\Delta+1}{2} \rfloor}$.*

In the 1980's, Kostochka proved the following using a complicated recoloring argument together with a technique for reducing Δ in a counterexample based on hitting every maximum clique with an independent set.

Theorem 1.4 (Kostochka [10]). *Every graph satisfying $\chi \geq \Delta$ contains a $K_{\Delta-28}$.*

Kostochka [10] proved the following result which shows that graphs having clique number sufficiently close to their maximum degree contain an independent set hitting every maximum clique. In [15] the second author improved the antecedent to $\omega \geq \frac{3}{4}(\Delta+1)$. Finally, King [9] made the result tight.

Lemma 1.5 (Kostochka [10]). *If G is a graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$, then G contains an independent set I such that $\omega(G-I) < \omega(G)$.*

Lemma 1.6 (Rabern [15]). *If G is a graph satisfying $\omega \geq \frac{3}{4}(\Delta+1)$, then G contains an independent set I such that $\omega(G-I) < \omega(G)$.*

Lemma 1.7 (King [9]). *If G is a graph satisfying $\omega > \frac{2}{3}(\Delta+1)$, then G contains an independent set I such that $\omega(G-I) < \omega(G)$.*

If G is a vertex critical graph satisfying $\omega > \frac{2}{3}(\Delta+1)$ and we expand the independent set I produced by Lemma 1.7 to a maximal independent set M and remove M from G, we see that $\Delta(G-M) \leq \Delta(G)-1$, $\chi(G-M) = \chi(G)-1$, and $\omega(G-M) = \omega(G)-1$. Using this, the proof of many coloring results can be reduced to the case of the smallest Δ for which they work. In the case of graphs with $\chi = \Delta$, we get the following general result.

Definition 1. For $k, j \in \mathbb{N}$, let $\mathcal{C}_{k,j}$ be the collection of all vertex critical graphs satisfying $\chi = \Delta = k$ and $\omega < k - j$. Put $\mathcal{C}_k := \mathcal{C}_{k,0}$. Note that $\mathcal{C}_{k,j} \subseteq \mathcal{C}_{k,i}$ for $j \geq i$.

For each $k \geq 9$, $\mathcal{C}_k$ is precisely the set of counterexamples to the Borodin-Kostochka Conjecture with $\Delta = k$.

Lemma 1.8. *Fix $k, j \in \mathbb{N}$ with $k \geq 3j+6$. If $G \in \mathcal{C}_{k,j}$, then there exists $H \in \mathcal{C}_{k-1,j}$ such that $H \lhd G$.*

Proof. Let $G \in \mathcal{C}_{k,j}$. We first show that there exists a maximal independent set M such that $\omega(G-M) < k-(j+1)$. If $\omega(G) < k-(j+1)$, then any maximal independent set will do for M. Otherwise, $\omega(G) = k-(j+1)$. Since $k \geq 3j+6$, we have $\omega(G) = k-(j+1) > \frac{2}{3}(k+1) = \frac{2}{3}(\Delta(G)+1)$. Thus by Lemma 1.7, we have an independent set I such that $\omega(G-I) < \omega(G)$. Expand I to a maximal independent set to get M.

Now $\chi(G-M) = k-1 = \Delta(G-M)$, where the last equality follows from Brooks' theorem and $\omega(G-M) < k-(j+1) \leq k-1$. Since $\omega(G-M) < k-(j+1)$, for any $(k-1)$-critical induced subgraph $H \unlhd G-M$ we have $H \in \mathcal{C}_{k-1,j}$. □

As a consequence we get the following result of Kostochka that the Borodin-Kostochka conjecture can be reduced to the case when $\Delta = k = 9$.

Lemma 1.9. *Let $\mathcal{H}$ be a hereditary graph property. For $k \geq 5$, if $\mathcal{H} \cap \mathcal{C}_k = \emptyset$, then $\mathcal{H} \cap \mathcal{C}_{k+1} = \emptyset$. In particular, to prove the Borodin-Kostochka conjecture it is enough to show that $\mathcal{C}_9 = \emptyset$.*

A little while after Kostochka proved his bound, Mozhan [13] proved the following using a different technique.

Theorem 1.10 (Mozhan [13])**.** *Every graph satisfying $\chi \geq \Delta \geq 10$ contains a $K_{\lfloor \frac{2\Delta+1}{3} \rfloor}$.*

In his dissertation Mozhan improved on this result. We don't know the method of proof as we were unable to obtain a copy of his dissertation. However, we suspect the method is a more complicated version of the proof of Theorem 1.10.

Theorem 1.11 (Mozhan)**.** *Every graph satisfying $\chi \geq \Delta \geq 31$ contains a $K_{\Delta-3}$.*

In 1999, Reed used probabilistic methods to prove that the Borodin-Kostochka conjecture holds for graphs with very large maximum degree.

Theorem 1.12 (Reed [16])**.** *Every graph satisfying $\chi \geq \Delta \geq 10^{14}$ contains a K_Δ.*

A lemma from Reed's proof of the above theorem is generally useful.

Lemma 1.13 (Reed [16])**.** *Let G be a vertex critical graph satisfying $\chi = \Delta \geq 9$ having the minimum number of vertices. If H is a $K_{\Delta-1}$ in G, then any vertex in $G - H$ has at most 4 neighbors in H. In particular, the $K_{\Delta-1}$'s in G are pairwise disjoint.*

1.2 Our contribution

We carry out an in-depth study of minimum counterexamples to the Borodin-Kostochka conjecture. Our main tool is the classification, in Section 4, of graph joins $A * B$ with $|A| \geq 2$, $|B| \geq 2$ which are f-choosable, where $f(v) := d(v) - 1$ for each vertex v. Since such a join cannot be an induced subgraph of a vertex critical graph with $\chi = \Delta$, we have a wealth of structural information about minimum counterexamples to the Borodin-Kostochka conjecture. In Section 2, we exploit this information and minimality to improve Reed's Lemma 1.13 as follows (see Corollary 2.11).

Lemma 1.14. *Let G be a vertex critical graph satisfying $\chi = \Delta \geq 9$ having the minimum number of vertices. If H is a $K_{\Delta-1}$ in G, then any vertex in $G - H$ has at most 1 neighbor in H.*

Moreover, we lift the result out of the context of a minimum counterexample to the Borodin-Kostochka conjecture, to the more general context of graphs satisfying a certain criticality condition—we call such graphs mules. This allows us to prove meaningful results for values of Δ less than 9.

Let K_t and E_t be the complete and edgeless graphs on t vertices, respectively. Since a graph containing K_Δ as a subgraph also contains $K_{t,\Delta-t}$ as a subgraph for any $t \in [\Delta - 1]$, the Borodin-Kosotochka conjecture implies the following conjecture. Our main result is that the two conjectures are equivalent.

Conjecture 1.15. *Any graph with $\chi = \Delta \geq 9$ contains some $A_1 * A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3$, $|A_1| + |A_2| = \Delta$ and $A_i \neq K_1 + K_{|A_i|-1}$ for some $i \in [2]$.*

In fact, using Kostochka's reduction (Lemma 1.9) to the case $\Delta = 9$, the following conjecture is also equivalent.

Conjecture 1.16. *Any graph with $\chi = \Delta = 9$ contains some $A_1 * A_2$ as an induced subgraph where $|A_1|, |A_2| \geq 3$, $|A_1| + |A_2| = 9$ and $A_i \neq K_1 + K_{|A_i|-1}$ for some $i \in [2]$.*

As a special case, we get a couple more palatable equivalent conjectures (see Lemma 2.18 and the comment following it).

Conjecture 1.17. *Any graph with $\chi = \Delta \geq 9$ contains $K_3 * E_{\Delta-3}$ as a subgraph.*

Conjecture 1.18. *Any graph with $\chi = \Delta = 9$ contains $K_3 * E_6$ as a subgraph.*

The condition $A_i \neq K_1 + K_{|A_i|-1}$ is unnatural and by removing it we get a (possibly) weaker conjecture than the Borodin-Kostochka conjecture which has more aesthetic appeal.

Conjecture 1.19. *Let G be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $3 \leq t \leq k-3$, then G can be $(k-1)$-colored.*

Conjecture 1.20. *Conjecture 1.19 is equivalent to the Borodin-Kostochka conjecture.*

Perhaps it would be easier to attack Conjecture 1.19 with $3 \leq t \leq k-3$ replaced by $2 \leq t \leq k-2$? We are unable to prove even this conjecture. Making this change and bringing k down to 5 gives the following conjecture, which, if true, would imply the remaining two cases of Grünbaum's girth problem for graphs with girth at least five.

Conjecture 1.21. *Let G be a graph with $\Delta(G) = k \geq 5$. If $K_{t,k-t} \not\subseteq G$ for all $2 \leq t \leq k-2$, then G can be $(k-1)$-colored.*

If G is a graph with with $\Delta(G) = k \geq 5$ and girth at least five, then it contains no $K_{t,k-t}$ for all $2 \leq t \leq k-2$ and hence Conjecture 1.21 would give a $(k-1)$-coloring. This conjecture would be tight since the Grünbaum graph and the Brinkmann graph are examples with $\chi = \Delta = 4$ and girth at least five.

Finally, we prove that the following conjecture is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6 (see Theorem 2.24).

Conjecture 1.22. *Every graph satisfying $\chi = \Delta = 9$ and $\alpha \leq 6$ contains a K_8.*

2 Mules

In this section we exclude more induced subgraphs in a minimum counterexample to the Borodin-Kostochka conjecture than we can exclude purely using list coloring properties. In fact, we lift these results out of the context of a minimum counterexample to graphs satisfying a certain criticality condition defined in terms of the following ordering.

Definition 2. If G and H are graphs, an *epimorphism* is a graph homomorphism $f: G \twoheadrightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 3. Let G be a graph. A graph A is called a *child* of G if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f: H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'. By definition, if $H \lhd G$ then $H \prec G$.

Lemma 2.1. *The ordering $\prec$ is well-founded on $\mathcal{G}$; that is, every nonempty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ minimizing $|G|$ and then maximizing $\|G\|$. Since any child of G must have fewer vertices or more edges (or both), we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. □

Definition 4. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a $\mathcal{T}$*-mule*.

With the definition of mule we have captured the important properties (for coloring) of a counterexample first minimizing the number of vertices and then maximizing the number of edges. Viewing $\mathcal{T}$ as a set of counterexamples, we can add edges to or contract independent sets in induced subgraphs of a $\mathcal{T}$-mule and get a non-counterexample. We could do the same with a minimal counterexample, but with mules we have more minimal objects to work with. One striking consequence of this is that many of our proofs naturally construct multiple counterexamples to Borodin-Kostochka for small Δ.

2.1 Excluding induced subgraphs in mules

Our main goal in this section is to prove Lemma 2.12, which says that (with only one exception) for $k \geq 7$, no k-mule contains $K_4 * E_{k-4}$ as a subgraph. This result immediately implies that the Borodin-Kostochka Conjecture is equivalent to Conjecture 2.13. This equivalence is a major step toward our main result. Our approach is based on Lemma 4.30, which implies that if G is a counterexample to Lemma 2.12, then the vertices of the E_{k-4} induce either E_3, a claw, a clique, or an almost complete graph. Our job in this section consists of showing that each of these four possibilities is, in fact, impossible. Ruling out the clique is easy. The cases of E_3 and the claw are handled in Lemma 2.8, and the case of an almost complete graph (which requires the most work) is handled by Corollary 2.11.

For $k \in \mathbb{N}$, by a *k-mule* we mean a $\mathcal{C}_k$-mule.

Lemma 2.2. *Let G be a k-mule with $k \geq 4$. If A is a child of G with $\Delta(A) \leq k$ then either*

- *A is $(k-1)$-colorable; or,*
- *A contains a K_k.*

Proof. Let A be a child of G with $\Delta(A) \leq k$, $H \trianglelefteq G$ and $f: H \twoheadrightarrow A$ an epimorphism. Without loss of generality, A is vertex critical. Suppose A is not $(k-1)$-colorable. Then $\chi(A) \geq k \geq \Delta(A)$. Since $A \prec G$ and G is a mule, $A \notin \mathcal{C}_k$. Thus we have $\chi(A) > \Delta(A) \geq 3$, so Brooks' theorem implies that $A = K_k$. □

Note that adding edges to a graph yields an epimorphism.

Lemma 2.3. *Let G be a k-mule with $k \geq 4$ and $H \trianglelefteq G$. Assume $x, y \in V(H)$, $xy \notin E(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$. If for every $(k-1)$-coloring π of H we have $\pi(x) = \pi(y)$, then H contains $\{x, y\} * K_{k-2}$.*

Proof. Suppose that for every $(k-1)$-coloring π of H we have $\pi(x) = \pi(y)$. Using the inclusion epimorphism $f_{xy}: H \twoheadrightarrow H + xy$ in Lemma 2.2 shows that either $H + xy$ is $(k-1)$-colorable or $H + xy$ contains a K_k. Since a $(k-1)$-coloring of $H + xy$ would induce a $(k-1)$-coloring of H with x and y colored differently, we conclude that $H + xy$ contains a K_k. But then H contains $\{x, y\} * K_{k-2}$ and the proof is complete. □

We will often begin by coloring some subgraph H of our graph G, and work to extend this partial coloring. More formally, let G be a graph and $H \triangleleft G$. For $t \geq \chi(H)$, let π be a proper t-coloring of H. For each $x \in V(G-H)$, put $L_\pi(x) := \{1, \ldots, t\} - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$. Then π is completable to a t-coloring of G iff L_π admits a coloring of $G - H$. We will use this fact repeatedly in the proofs that follow. The following generalizes a lemma due to Reed [16], the proof is essentially the same.

Lemma 2.4. *For $k \geq 6$, if a k-mule G contains an induced $E_2 * K_{k-2}$, then G contains an induced $E_3 * K_{k-2}$.*

Proof. Suppose G is a k-mule containing an induced $E_2 * K_{k-2}$, call it F. Let x, y be the vertices of degree $k-2$ in F and $C := \{w_1, \ldots, w_{k-2}\}$ the vertices of degree $k-1$ in F. Put $H := G - F$. Since G is vertex critical, we may $k-1$ color H. Doing so leaves a list assignment L on F with $|L(z)| \geq d_F(z) - 1$ for each $z \in V(F)$. Now $|L(x)| + |L(y)| \geq d_F(x) + d_F(y) - 2 = 2k - 6 > k - 1$ since $k \geq 6$. Hence we have $c \in L(x) \cap L(y)$. Coloring both x and y with c leaves a list assignment L' on C with $|L'(w_i)| \geq k-3$ for each $1 \leq i \leq k-2$. Now, if $|L'(w_i)| \geq k-2$ or $L'(w_i) \neq L'(w_j)$ for some i, j, then we can complete the partial $(k-1)$-coloring to all of G using Hall's Theorem. Hence we must have $d(w_i) = k$ and $L'(w_i) = L'(w_j)$ for all i, j. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that N is an independent set since it is contained in a single color class in every $(k-1)$-coloring of H. Also, each $w \in C$ has exactly one neighbor in N.

Proving that $|N| = 1$ will give the desired $E_3 * K_{k-2}$ in G. Thus, to reach a contradiction, suppose that $|N| \geq 2$.

We know that H has no $(k-1)$-coloring in which two vertices of N get different colors since then we could complete the partial coloring as above. Let $v_1, v_2 \in N$ be different. Since both v_1 and v_2 have a neighbor in F, we may apply Lemma 2.3 to conlcude that $\{v_1, v_2\} * K_{v_1,v_2}$ is in H, where K_{v_1,v_2} is a K_{k-2}.

First, suppose $|N| \geq 3$, say $N = \{v_1, v_2, v_3\}$. We have $z \in K_{v_1,v_2} \cap K_{v_1,v_3}$ for otherwise $d(v_1) \geq 2(k-2) > k$. Since z already has k neighbors among $K_{v_1,v_2} - \{z\}$ and v_1, v_2, v_3, we must have $K_{v_1,v_3} = K_{v_1,v_2}$. But then $\{v_1, v_2, v_3\} + K_{v_1,v_2}$ is our desired $E_3 * K_{k-2}$ in G.

Hence we must have $|N| = 2$, say $N = \{v_1, v_2\}$. For $i \in [2]$, v_i has $k-2$ neighbors in K_{v_1,v_2} and thus at most two neighbors in C. Hence $|C| \leq 4$. Thus we must have $k = 6$.

We may apply the same reasoning to $\{v_1, v_2\} * K_{v_1,v_2}$ that we did to F to get vertices $v_{2,1}, v_{2,2}$ such that $\{v_{2,1}, v_{2,2}\} * K_{v_{2,1},v_{2,2}}$ is in G. But then we may do it again with $\{v_{2,1}, v_{2,2}\} * K_{v_{2,1},v_{2,2}}$ and so on. Since G is finite, at some point this process must terminate. But the only way to terminate is to come back around and use x and y. This graph is 5-colorable since we may color all the E_2's with the same color and then 4-color the remaining K_4 components. This final contradiction completes the proof. □

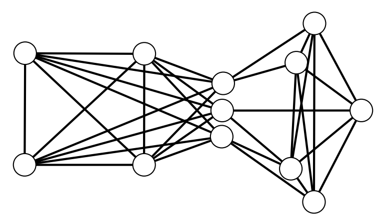


Figure 1: The mule $M_{6,1}$.

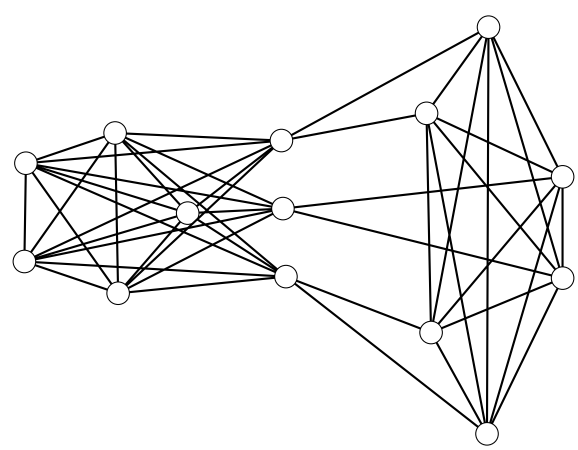


Figure 2: The mule $M_{7,1}$.

Lemma 2.5. *For $k \geq 6$, the only k-mules containing an induced $E_2 * K_{k-2}$ are $M_{6,1}$ and $M_{7,1}$.*

Proof. Suppose we have a k-mule G that contains an induced $E_2 * K_{k-2}$. Then by Lemma 2.4, G contains an induced $E_3 * K_{k-2}$, call it F.

Let x, y, z be the vertices of degree $k-2$ in F and let $C := \{w_1, \ldots, w_{k-2}\}$ be the vertices of degree k in F. Put $H := G - C$. Since each of x, y, z have degree at most 2 in H and G is a mule, the homomorphism from H sending x, y, and z to the same vertex must produce a K_k. Thus we must have $k \leq 7$ and H contains a K_{k-1} (call it D) such that $V(D) \subseteq N(x) \cup N(y) \cup N(z))$. Put $A := G[V(F) \cup V(D)]$. Then A is k-chromatic and as G is a mule, we must have $G = A$. If $k = 7$, then $G = M_{7,1}$. Suppose $k = 6$ and $G \neq M_{6,1}$. Then one of x, y, or z has only one neighbor in D. By symmetry we may assume it is x. But we can add an edge from x to a vertex in D to form $M_{6,1}$ and hence G has a proper child, which is impossible. □

Lemma 2.6. *Let G be a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $H \lhd G$. If $x, y \in V(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$, then there exists a $(k-1)$-coloring π of H such that $\pi(x) \neq \pi(y)$.*

Proof. Suppose $x, y \in V(H)$ and both $d_H(x) \leq k-1$ and $d_H(y) \leq k-1$. First, if $xy \in E(H)$ then any $(k-1)$-coloring of H will do. Otherwise, if for every $(k-1)$-coloring π of H we have $\pi(x) = \pi(y)$, then by Lemma 2.3, H contains $\{x, y\} * K_{k-2}$. The lemma follows since this is impossible by Lemma 2.5. □

Lemma 2.7. *Let G be a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$ and let $F \lhd G$. Put $C := \{v \in V(F) \mid d(v) - d_F(v) \leq 1\}$. At least one of the following holds:*

- *$G - F$ has a $(k-1)$-coloring π such that for some $x, y \in C$ we have $L_\pi(x) \neq L_\pi(y)$; or,*
- *$G - F$ has a $(k-1)$-coloring π such that for some $x \in C$ we have $|L_\pi(x)| = k-1$; or,*
- *there exists $z \in V(G-F)$ such that $C \subseteq N(z)$.*

Proof. Put $H := G-F$. Suppose that for every $(k-1)$-coloring π of H we have $L_\pi(x) = L_\pi(y)$ for every $x, y \in C$. By assumption, the vertices in C have at most one neighbor in H. If some $v \in C$ has no neighbors in H, then for any $(k-1)$-coloring π of H we have $|L_\pi(v)| = k-1$. Thus we may assume that every $v \in C$ has exactly one neighbor in H.

Let $N := \bigcup_{w \in C} N(w) \cap V(H)$. Suppose $|N| \geq 2$. Pick different $z_1, z_2 \in N$. Then, by Lemma 2.6, there is a $(k-1)$-coloring π of H for which $\pi(z_1) \neq \pi(z_2)$. But then $L_\pi(x) \neq L_\pi(y)$ for some $x, y \in C$ giving a contradiction. Hence $N = \{z\}$ and thus $C \subseteq N(z)$. □

By Lemma 4.24, no graph in $\mathcal{C}_k$ contains an induced $E_3 * K_{k-3}$ for $k \geq 9$. For mules, we can improve this as follows.

Lemma 2.8. *For $k \geq 7$, the only k-mule containing an induced $E_3 * K_{k-3}$ is $M_{7,1}$.*

Proof. Suppose the lemma is false and let G be a k-mule, other than $M_{7,1}$, containing such an induced subgraph F. Let $z_1, z_2, z_3 \in F$ be the vertices with degree $k-3$ in F and C the rest of the vertices in F (all of degree $k-1$ in F). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of C. Let π be a $(k-1)$-coloring of H guaranteed by Lemma 2.7 and put $L := L_\pi$. Since $|L(z_1)| + |L(z_2)| + |L(z_3)| \geq 3(k-4) > k-1$ we have $1 \leq i < j \leq 3$ such that $L(z_i) \cap L(z_j) \neq \emptyset$. Without loss of generality, $i = 1$ and $j = 2$. Pick $c \in L(z_1) \cap L(z_2)$ and color both z_1 and z_2 with c. Let L' be the resulting list assignment on $F - \{z_1, z_2\}$. Now $|L'(z_3)| \geq k-4$ and $|L'(v)| \geq k-3$ for each $v \in C$. By our choice of π, either two of the lists in C differ or for some $v \in C$ we have $|L'(v)| \geq k-2$. In either case, we can complete the $(k-1)$-coloring to all of G by Hall's Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of C. Thus G contains the induced subgraph $K_{k-3} * G[z_1, z_2, z_3, x]$. Therefore $k = 7$ and x is adjacent to each of z_1, z_2, z_3 by Lemma 4.30. Hence G contains the induced subgraph $K_5 * E_3$ contradicting Lemma 2.5. □

Lemma 2.9. *For $k \geq 7$, no k-mule contains an induced $\overline{P_3} * K_{k-3}$.*

Proof. Suppose the lemma is false and let G be a k-mule containing such an induced subgraph F. Note that $M_{7,1}$ has no induced $\overline{P_3} * K_{k-3}$, so $G \neq M_{7,1}$. Let $z \in V(F)$ be the vertex with degree $k-3$ in F, $v_1, v_2 \in F$ the vertices of degree $k-2$ in F and C the rest of the vertices in F (all of degree $k-1$ in F). Put $H := G - F$.

First suppose there is not a vertex $x \in V(H)$ which is adjacent to all of C. Let π be a $(k-1)$-coloring of H guaranteed by Lemma 2.7 and put $L := L_\pi$. Then, we have $|L(z)| \geq k-4$ and $|L(v_1)| \geq k-3$. Since $k \geq 7$, $|L(z)| + |L(v_1)| \geq 2k-7 > k-1$. Hence, by Lemma 4.7, we may color z and v_1 the same. Let L' be the resulting list assignment on $F - \{z, v_1\}$. Now $|L'(v_2)| \geq k-4$ and $|L'(v)| \geq k-3$ for each $v \in C$. By our choice of π, either two of the lists in C differ or for some $v \in C$ we have $|L'(v)| \geq k-2$. In either case, we can complete the $(k-1)$-coloring to all of G by Hall's Theorem.

Hence we must have $x \in V(H)$ which is adjacent to all of C. Thus G contains the induced subgraph $K_4 * G[z, v_1, v_2, x]$. By Lemma 4.30, $G[z, v_1, v_2, x]$ must be almost complete and hence x must be adjacent to both v_1 and v_2. But then $G[v_1, v_2, x] * C$ is a K_k in G, giving a contradiction. □

Reed proved that for $k \geq 9$, a vertex outside a $(k-1)$-clique H in a k-mule can have at most 4 neighbors in H. We improve this to at most one neighbor.

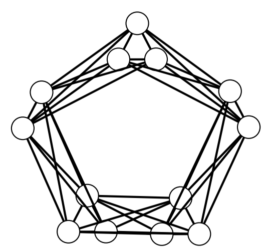


Figure 3: The mule $M_{7,2}$.

Lemma 2.10. *For $k \geq 7$ and $r \geq 2$, no k-mule except $M_{7,1}$ and $M_{7,2}$ contains an induced $K_r * \left(K_1 + K_{k-(r+1)}\right)$.*

Proof. Suppose the lemma is false and let G be a k-mule, other than $M_{7,1}$ and $M_{7,2}$, containing such an induced subgraph F with r maximal. By Lemma 2.5 and Lemma 2.9, the lemma holds for $r \geq k-3$. So we have $r \leq k-4$. Now, let $z \in V(F)$ be the vertex with degree r in F, $v_1, v_2, \ldots, v_{k-(r+1)} \in V(F)$ the vertices of degree $k-2$ in F and C the rest of the vertices in F (all of degree $k-1$ in F). Put $H := G - F$.

Let $Z_1 := \{za \mid a \in N(v_1) \cap V(H)\}$. Consider the graph $D := H + z + Z_1$. Since v_1 has at most two neighbors in H, $|Z_1| \leq 2$ and thus to form D from $H + z$, we added $E(A)$ where $A \in \{K_1, K_2, P_3\}$. Since $|C| \geq 2$, $\Delta(D) \leq k$. Hence Lemma 2.2 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k-1$. Suppose $\chi(D) \geq k$. If $A = K_1$, $A = K_2$, or $A = P_3$, then we have a contradiction by the fact that $\omega(G) < k$, Lemma 2.5, and Lemma 2.9, respectively. Thus we must have $\chi(D) \leq k-1$, which gives a $(k-1)$-coloring of $H + z$ in which z receives a color c which is not received by any of the neighbors of v_1 in H. Thus c remains in the list of v_1 and we may color v_1 with c. After doing so, each vertex in C has a list of size at least $k-3$ and v_i for $i > 1$ has a list of size at least $k-4$. If any pair of vertices in

C had different lists, then we could complete the partial coloring by Hall's Theorem. Let $N := \bigcup_{w \in C} N(w) \cap V(H)$ and note that N is an independent set since it is contained in a single color class in the $(k-1)$-coloring of H just constructed.

Suppose $|N| \geq 2$. Pick $a_1, a_2 \in N$. Consider the graph $D := H + z + Z_1 + a_1a_2$. Plainly, $\Delta(D) \leq k$. To form D from $H + z$ we added $E(A)$, where $A \in \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3\}$. Hence Lemma 2.2 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$. If $\chi(D) \geq k$, then we have a contradiction since $A = K_1$, $A = K_2$, and $A = P_3$ are impossible as above. To show that $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible, we apply Lemma 2.8 (this is where we use the fact that $G \neq M_{7,1}$), Lemma 4.32 (since $K_t - E(P_4) = P_4 * K_{t-4}$), and Lemma 4.27, respectively.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k-1)$-coloring of $H + z$ in which a_1 and a_2 are in different color classes and z receives a color not received by any neighbor of v_1 in H. As above we can complete this partial coloring to all of G by first coloring z and v_1 the same and then using Hall's Theorem.

Hence there is a vertex $x \in V(H)$ which is adjacent to all of C. Note that x is not adjacent to any of $v_1, v_2, \ldots, v_{k-(r+1)}$ by the maximality of r. Let $Z_2 := \{xa \mid a \in N(v_2) \cap V(H)\}$. Consider the graph $D := H + z + Z_1 + Z_2$. As above, both Z_1 and Z_2 have cardinality at most 2. Since $|C| \geq 2$, both x and z have degree at most k in D. Since both xa and za were added only if a was a neighbor of both v_1 and v_2, all the neighbors of v_1 in H have degree at most k in D. Similarly for v_2's neighbors. Hence $\Delta(D) \leq k$. To form D from $H + z$ we added $E(A)$ where $A \in \{K_1, K_2, P_3, K_3, P_4, K_2 + P_3, 2K_2, P_5, 2P_3, C_4\}$. Hence Lemma 2.2 shows that $H + z$ contains a $K_k - E(A)$ or $\chi(D) \leq k - 1$.

Suppose $\chi(D) \geq k$. Then $A = K_1$, $A = K_2$, $A = P_3$, $A = K_3$, $A = P_4$, and $A = K_2 + P_3$ are impossible as above. Applying Lemma 4.27 shows that $A = 2K_2$, $A = P_5$, and $A = 2P_3$ are impossible. Thus we must have $A = C_4$. If $k \geq 8$, then Lemma 4.23 gives a contradiction. Hence we must have $k = 7$. Since $H + z$ contains an induced $K_3 * 2K_2$, we must have $N(v_1) \cap V(H) = N(v_2) \cap V(H)$, say $N(v_1) \cap V(H) = \{w_1, w_2\}$. Moreoever, $xz \in E(G)$, $w_1w_2 \in E(G)$ and there are no edges between $\{w_1, w_2\}$ and $\{x, z\}$ in G.

Put $Q := \{v_1, \ldots, v_{k-(r+1)}\}$. Then for $v \in Q$, by the same argument as above, we must have $N(v) \cap V(H) = \{w_1, w_2\}$. Hence Q is joined to $\{w_1, w_2\}$, C is joined to Q, and $\{x, z\}$ and both $\{x, z\}$ and $\{w_1, w_2\}$ are joined to the same K_3 in H. We must have $r = 3$ for otherwise one of x, z, w_1, w_2 has degree larger than 7. Thus we have an $M_{7,2}$ in G and therefore G is $M_{7,2}$, a contradiction.

Thus we must have $\chi(D) \leq k - 1$, which gives a $(k-1)$-coloring of $H + z$ in which z receives a color c_1 which is not received by any of the neighbors of v_1 in H and x receives a color c_2 which is not received by any of the neighbors of v_2 in H. Thus c_1 is in v_1's list and c_2 is in v_2's list. Note that if x and z are adjacent then $c_1 \neq c_2$. Hence, we can 2-color $G[x, z, v_1, v_2]$ from the lists. This leaves $k - 3$ vertices. The vertices in C have lists of size at least $k - 3$ and the rest have lists of size at least $k - 5$. Since the union of any $k - 4$ of the lists contains one list of size $k - 3$, we can complete the partial coloring by Hall's Theorem. □

Corollary 2.11. *For $k \geq 7$, if H is a $(k-1)$-clique in a k-mule G other than $M_{7,1}$ and $M_{7,2}$, then any vertex in $G - H$ has at most one neighbor in H.*

Proof. Let $v \notin H$ be adjacent to r vertices in H. Now $G[H \cup \{v\}] = K_r * (K_1 + K_{k-(r+1)})$. If $r \geq 2$, then $G[H \cup \{v\}]$ is forbidden by Lemma 2.10. □

Lemma 2.12. *For $k \geq 7$, no k-mule except $M_{7,1}$ contains $K_4 * E_{k-4}$ as a subgraph.*

Proof. Let G be a k-mule other than $M_{7,1}$ and suppose G contains an induced $K_4 * D$ where $|D| = k - 4$. Then G is not $M_{7,2}$. By Lemma 4.30, D is E_3, a claw, a clique, or almost complete. If D is a clique then G contains K_k, a contradiction. Now Corollary 2.11 shows that D being almost complete is impossible. Finally, Lemma 2.8 shows that D cannot be E_3 or a claw. This contradiction completes the proof. □

Since $K_4 * E_{\Delta-4} \subseteq K_\Delta$, Lemma 2.12 shows that the following conjecture is equivalent to the Borodin-Kostochka conjecture.

Conjecture 2.13. *Any graph with $\chi \geq \Delta \geq 9$ contains $K_4 * E_{\Delta-4}$ as a subgraph.*

Lemma 2.14. *Let G be a k-mule with $k \geq 8$. Let A and B be graphs with $4 \leq |A| \leq k-4$ and $|B| = k - |A|$ such that $A * B \trianglelefteq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

Proof. Note that $|B| \geq 4$. By Lemma 4.49, $A * B$ is almost complete, $K_5 * E_3$ or our desired conclusion holds. The first and second cases are impossible by Corollary 2.11 and Lemma 2.8. □

This shows that the following conjecture is a natural weakening of Borodin-Kostochka.

Conjecture 2.15. *Let G be a graph with $\Delta(G) = k \geq 9$. If $K_{t,k-t} \not\subseteq G$ for all $4 \leq t \leq k-4$, then G can be $(k-1)$-colored.*

In the next section we create the tools needed to reduce the 4 in these lemmata to 3.

2.2 Tooling up

For an independent set I in a graph G, we write $\frac{G}{[I]}$ for the graph formed by collapsing I to a single vertex and discarding duplicate edges. We write $[I]$ for the resulting vertex in the new graph. If more than one independent set $I_1, I_2, \ldots, I_m$ are collapsed in succession we indicate the resulting graph by $\frac{G}{[I_1][I_2]\cdots[I_m]}$.

Lemma 2.16. *Let G be a k-mule other than $M_{7,1}$ and $M_{7,2}$ with $k \geq 7$ and $H \lhd G$. If $x, y \in V(H)$, $xy \notin E(H)$ and $|N_H(x) \cup N_H(y)| \leq k$, then there exists a $(k-1)$-coloring π of H such that $\pi(x) = \pi(y)$.*

Proof. Suppose $x, y \in V(H)$, $xy \notin E(H)$ and $|N_H(x) \cup N_H(y)| \leq k$. Put $H' := \frac{H}{[x,y]}$. Then $H' \prec H$ via the natural epimorphism $f \colon H \twoheadrightarrow H'$. By applying Lemma 2.2 we either get the desired $(k-1)$-coloring π of H or a K_{k-1} in H with $V(K_{k-1}) \subseteq N(x) \cup N(y)$. But $k - 1 \geq 6$, so one of x or y has at least three neighbors in K_{k-1} violating Corollary 2.11. □

Lemma 2.17. *Let G be a k-mule other than $M_{7,1}$ and $M_{7,2}$ with $k \geq 7$ and $H \lhd G$. Suppose there are disjoint nonadjacent pairs $\{x_1, y_1\}, \{x_2, y_2\} \subseteq V(H)$ with $d_H(x_1), d_H(y_1) \leq k-1$ and $|N_H(x_2) \cup N_H(y_2)| \leq k$. Then there exists a $(k-1)$-coloring π of H such that $\pi(x_1) \neq \pi(y_1)$ and $\pi(x_2) = \pi(y_2)$.*

Proof. Put $H' := \frac{H}{[x_2,y_2]} + x_1y_1$. Then $H' \prec H$ via the natural epimorphism $f\colon H \twoheadrightarrow H'$. Suppose the desired $(k-1)$-coloring π of H doesn't exist. Apply Lemma 2.2 to get a K_k in H'. Put $z := [x_2, y_2]$. By Lemma 2.5 the K_k must contain z and by Lemma 2.10, the K_k must contain x_1y_1; hence the K_k contains x_1, y_1, and z. Thus H contains an induced subgraph $A := \{x_1, y_1\} * K_{k-3}$ where $V(A) \subseteq N_H(x_2) \cup N_H(y_2)$. Then x_2 and y_2 each have at most two neighbors in the K_{k-3} by Lemma 2.12 and Lemma 4.34. Thus $k = 7$ and both x_2 and y_2 have exactly two neighbors in the K_4. One of x_2 or y_2 has at least one neighbor in $\{x_1, y_1\}$, so by symmetry we may assume that x_2 is adjacent to x_1. But then $\{x_2\} \cup V(A)$ induces either a $K_2 *$ antichair (if $x_2 \not\leftrightarrow y_1$) or a graph containing $K_2 * C_4$ (if $x_2 \leftrightarrow y_1$), and both are impossible by Lemma 4.50. □

2.3 Using our new tools

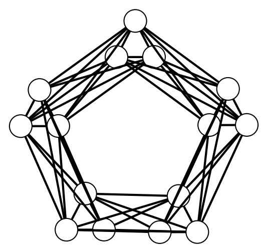


Figure 4: The mule M_8.

Lemma 2.18. *For $k \geq 7$, the only k-mules containing $K_3 * E_{k-3}$ as a subgraph are $M_{7,1}$, $M_{7,2}$ and M_8.*

Proof. Suppose not and let G be a k-mule other than $M_{7,1}$, $M_{7,2}$ and M_8 containing $F := C * B$ as an induced subgraph where $C = K_3$ and B is an arbitrary graph with $|B| = k - 3$. By Lemma 4.34, B is: $E_3 * K_{|B|-3}$, almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, or $E_3 + K_{|B|-3}$. The first two options are impossible by Lemma 2.12.

First, suppose there is no $z \in V(G - F)$ with $C \subseteq N(z)$. Let π be the $(k-1)$-coloring of $G - F$ guaranteed by Lemma 2.7. Put $L := L_\pi$. Let I be a maximal independent set in B. If there are $x, y \in I$ and $c \in L_(x) \cap L_(y)$, then we may color x and y with c and then greedily complete the coloring to the rest of F giving a contradiction. Thus we must have

$$
\begin{aligned}
k-1 &\geq \sum_{v\in I} |L(v)| \\
&\geq \sum_{v\in I} (d_F(v)-1) \\
&= \sum_{v\in I} (d_B(v)+3-1) \\
&= 2|I| + \sum_{v\in I} d_B(v) \\
&= |B| + |I| \\
&= k-3+|I|.
\end{aligned}
$$

Therefore $|I| \leq 2$ and hence B is $K_t + K_{|B|-t}$. Put $N := \bigcup_{w\in C} N(w) \cap V(G-F)$. Then $|N| \geq 2$ by assumption. Pick $x_1, y_1 \in N$ and nonadjacent $x_2, y_2 \in V(B)$ and put $H := G[V(G-F) \cup \{x_2, y_2\}]$. Plainly, the conditions of Lemma 2.17 are satisfied and hence we have a $(k-1)$-coloring γ of H such that $\gamma(x_1) \neq \gamma(y_1)$ and $\gamma(x_2) = \gamma(y_2)$. But then we can greedily complete this coloring to all of G, a contradiction.

Thus we have $z \in V(G-F)$ with $C \subseteq N(z)$. Put $B' := G[V(B) \cup \{z\}]$ and $F' := G[V(F) \cup \{z\}]$. As above, using Lemma 4.34 and Lemma 2.12, we see that B' is $K_t + K_{|B'|-t}$, $K_1 + K_t + K_{|B'|-t-1}$ or $E_3 + K_{|B'|-3}$.

Suppose B' is $E_3 + K_{|B'|-3}$, say the E_3 is $\{z_1, z_2, z_3\}$. Since $k \geq 7$, we have $w_1, w_2 \in V(B') - \{z_1, z_2, z_3\}$. Then $d_{F'}(z_3) + d_{F'}(w_1) = k$ and hence we may apply Lemma 2.16 to get a $(k-1)$-coloring ζ of $G - F'$ such that there is some $c \in L_\zeta(z_3) \cap L_\zeta(w_1)$. Now $|L_\zeta(z_1)| + |L_\zeta(z_2)| + |L_\zeta(w_2)| \geq 2 + 2 + k - 4 = k$ and hence there is a color c_1 that is in at least two of $L_\zeta(z_1)$, $L_\zeta(z_2)$ and $L_\zeta(w_2)$. If $c_1 = c$, then c appears on an independent set of size 3 in B' and we may color this set with c and greedily complete the coloring. Otherwise, B' contains two disjoint nonadjacent pairs which we can color with different colors and again complete the coloring greedily, a contradiction.

Now suppose B' is $K_1 + K_t + K_{|B'|-t-1}$. By Lemma 2.10, we must have $2 \leq t \leq |B'| - 3$. Let x be the vertex in the K_1, $w_1, w_2 \in V(K_t)$ and $z_1, z_2 \in V(K_{|B'|-t-1})$. Then $d_{F'}(w_1) + d_{F'}(z_1) = k+1$ and hence we may apply Lemma 2.16 to get a $(k-1)$-coloring ζ of $G-F'$ such that there is some $c \in L_\zeta(w_1) \cap L_\zeta(z_1)$. Now $|L_\zeta(x)| + |L_\zeta(w_2)| + |L_\zeta(z_2)| \geq 2 + k - 1 = k + 1$ and hence there is are at least two colors c_1, c_2 that are each in at least two of $L_\zeta(x)$, $L_\zeta(w_2)$ and $L_\zeta(z_2)$. If $c_1 \neq c$ or $c_2 \neq c$, then B' contains two disjoint nonadjacent pairs which we can color with different colors and then complete the coloring greedily. Otherwise c appears on an independent set of size 3 in B' and we may color this set with c and greedily complete the coloring, a contradiction.

Therefore B' must be $K_t + K_{|B'|-t}$. By Lemma 2.10, we must have $3 \leq t \leq |B'| - 3$. Thus $k \geq 8$. Let X and Y be the two cliques covering B'. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Put $H := G[V(G-F') \cup \{x_1, x_2, y_1, y_2\}]$ and $H' := \frac{H}{[x_1,y_1][x_2,y_2]}$. For $i \in [2]$, $d_{F'}(x_i) + d_{F'}(y_i) = k+2$ and thus $\Delta(H') \leq k$. If $\chi(H') \leq k-1$, then we have a $(k-1)$-coloring of H which can be greedily completed to all of G, a contradiction. Hence, by Lemma 2.2, H' contains

K_k. Thence $H - \{x_1, y_1, x_2, y_2\}$ contains a K_{k-2}, call it A, such that $V(A) \subseteq N(x_i) \cup N(y_i)$ for $i \in [2]$. Since $d_{F'}(x_i) + d_{F'}(y_i) = k + 2$, we see that $N_H(x_i) \cap N_H(y_i) = \emptyset$ for $i \in [2]$. But we can play the same game with the pairs $\{x_1, y_2\}$ and $\{x_2, y_1\}$. We conclude that $N(x_1) \cap V(A) = N(x_2) \cap V(A)$ and $N(y_1) \cap V(A) = N(y_2) \cap V(A)$. In fact we can extend this equality to all of X and Y. Put $Q := N(x_1) \cap V(A)$ and $P := N(y_1) \cap V(A)$. Then we conclude that X is joined to Q and Y is joined to P. Moreover, we already know that X and Y are joined to the same K_3. The edges in these joins exhaust the degrees of all the vertices, hence G is a 5-cycle with vertices blown up to cliques. If $k = 8$, then $|X| = |Y| = 3$ and thus $|Q| = |P| = 3$, but then $G = M_8$, a contradiction. So $k \geq 9$, but now G is a line graph of a multigraph, so this is impossible by the Borodin-Kostochka conjecture for line graphs proved in [14]. □

Since $K_3 * E_{\Delta-3} \subseteq K_\Delta$, Lemma 2.18 shows that Conjecture 1.17 is equivalent to the Borodin-Kostochka conjecture.

Lemma 2.19. *Let G be a k-mule with $k \geq 7$ other than $M_{7,1}$, $M_{7,2}$ and M_8. Let A and B be graphs with $3 \leq |A| \leq k - 3$ and $|B| = k - |A|$ such that $A * B \trianglelefteq G$. Then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

Proof. Suppose the lemma is false and let $A * B \trianglelefteq G$ be a counterexample.

First suppose $|A|, |B| \geq 4$. Then, by Lemma 4.49, $A * B$ is almost complete or $K_5 * E_3$. The first and second cases are impossible by Corollary 2.11 and Lemma 2.8 respectively.

Thus we may assume $|A| = 3$. By Lemma 2.18, $A \in \{E_3, P_3, K_1 + K_2\}$. If $A = E_3$, then B is complete by Lemma 4.44, but this is impossible by Lemma 2.8. If $A = P_3$, then B is complete by Lemma 4.25, but this is impossible by Lemma 2.5. Hence $A = K_1 + K_2$. By Lemma 4.48, B is complete or $K_1 + K_{|B|-1}$. The former is impossible by Lemma 2.9 and the latter by supposition. □

Lemma 2.19 proves our main result, that Conjecture 1.15 is equivalent to the Borodin-Kostochka conjecture.

2.4 The low vertex subgraph of a mule

In this section we show that if a mule is not regular, then the subgraph of non-maximum-degree vertices is severely restricted. For a vertex critical graph G we write $\mathcal{L}(G)$ for the subgraph induced on the vertices of degree $\chi(G) - 1$ in G and $\mathcal{H}(G)$ for the subgraph induced on the rest of the vertices. We call $v \in V(G)$ *low* if $v \in V(\mathcal{L}(G))$ and *high* otherwise.

Lemma 2.20. *For $k \geq 6$, no k-mule contains an induced $E_2 * K_{k-2}$ with some vertex low.*

Proof. Since $M_{6,1}$ and $M_{7,1}$ contain no such induced subgraph, the lemma follows from Lemma 2.5. □

Lemma 2.21. *If G is a k-mule with $k \geq 6$, then $\mathcal{L}(G)$ is complete.*

Proof. Let G be a k-mule with $k \geq 6$ and suppose G has nonadjacent low vertices x and y. Then $G + xy \prec G$ and hence, by Lemma 2.2, $G + xy$ contains a K_k. But then G contains an $E_2 * K_{k-2}$ with some vertex low, contradicting Lemma 2.20. Hence $\mathcal{L}(G)$ is complete. □

Lemma 2.22. *If G is a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$, then $|\mathcal{L}(G)| \leq k-2$.*

Proof. Let G be a k-mule with $k \geq 6$ other than $M_{6,1}$ and $M_{7,1}$. By Lemma 2.21, $\mathcal{L}(G)$ is complete and hence $|\mathcal{L}(G)| \leq k-1$. Suppose $|\mathcal{L}(G)| = k-1$. Since G doesn't contain K_k, no high z is adjacent to all of $\mathcal{L}(G)$. Hence, by Lemma 2.7, there is a $(k-1)$-coloring of $\mathcal{H}(G)$ that we can complete to all of G using Hall's Theorem. This contradiction completes the proof. □

Lemma 2.23. *Let G be a k-mule with $k \geq 6$. If a high $x \in V(G)$ has at least three low neighbors, then x is adjacent to all low vertices in G.*

Proof. Assume the lemma is false. Let x be a high degree vertex with at least three neighbors in $V(\mathcal{L}(G))$. If $|V(\mathcal{L}(G))| = 3$, then the claim holds. So assume that $|V(\mathcal{L}(G))| \geq 4$ and choose $y \in V(\mathcal{L}(G)) \setminus N(x)$. Let $A = V(\mathcal{L}(G)) \cap N(x)$. By Lemma 2.21, $\mathcal{L}(G)$ is complete. Thus, $G[\{x,y\} \cup A] = E_2 * K_{|A|}$. Since $L(v) = d(v)$ for all $v \in (A \cup \{y\})$, Lemma 4.54 implies that $E_2 * K_{|A|}$ cannot appear in G. This contradiction implies the lemma. □

2.5 Restrictions on the independence number

The Borodin-Kostochka conjecture has been proven for graphs with independence number at most two [1]. Here we prove that if we wish to prove the Borodin-Kostochka conjecture for graphs with independence number at most a for any $a \leq 6$, it suffices to construct a $K_{\Delta-1}$.

For $a \geq 2$, let $\mathcal{C}_k^a$ be those $G \in \mathcal{C}_k$ with $\alpha(G) \leq a$. By a (k,a)-mule we mean a $\mathcal{C}_k^a$-mule. Note that if $G \in \mathcal{C}_k^a$ and for some $H \in \mathcal{C}_k$ we have $H \prec G$, then $H \in \mathcal{C}_k^a$ as well. Therefore any (k,a)-mule is also a k-mule.

Theorem 2.24. *For $k \geq 7$ and $2 \leq a \leq k-3$, no (k,a)-mule except $M_{7,1}$ contains a K_{k-1}.*

Proof. Suppose otherwise and let G be such a (k,a)-mule containing a K_{k-1}, call it H. By Corollary 2.11, each vertex in $G-H$ has at most one neighbor in H. Let π be a $(k-1)$-coloring of $G-H$. Then $|L_\pi(v)| \geq k-3$ for all $v \in V(H)$. Since H cannot be colored from L_π, applying Hall's Theorem shows that either $|Pot(L_\pi)| \leq k-2$ or there is some $x \in V(H)$ such that $|Pot_{H-x}(L_\pi)| \leq k-3$. In the former case, π must have some color class to which each vertex of H is adjacent and hence $\alpha(G) \geq k-1$, a contradiction. In the latter case, π must have two color classes to which each vertex of $H-x$ is adjacent and hence G has two disjoint independent sets of size $k-2$. Again we have a contradiction since $\alpha(G) \geq k-2$. □

It follows that Conjecture 1.22 is equivalent to the Borodin-Kostochka conjecture for graphs with independence number at most 6.

3 Connectivity of complements

As a basic application of our list coloring lemmas, we prove that for $k \geq 5$ any $G \in \mathcal{C}_k$ has maximally connected complement.

Lemma 3.1. *Fix $k \geq 5$. If $G \in \mathcal{C}_k$ and $A * B \lhd G$ for graphs A and B with $1 \leq |A| \leq |B|$, then $|A * B| \leq \Delta(G) + 1$.*

Proof. Let $G \in \mathcal{C}_k$ and $A * B \unlhd G$ for graphs A and B with $1 \leq |A| \leq |B|$. Assume $|A * B| > \Delta(G) + 1$. To avoid a vertex with degree larger than $\Delta(G)$, we must have $\Delta(A) \leq |A| - 2$ and $\Delta(B) \leq |B| - 2$. In particular, both A and B are incomplete, so $2 \leq |A| \leq |B|$ and both A and B contain an induced E_2. Hence, by Lemma 4.27, both A and B are the disjoint union of complete subgraphs and at most one P_3.

First, assume $|A| = 2$, say $A = \{x_1, x_2\}$. Since $|B| \geq \Delta(G)$, we conclude that $N(x_1) = N(x_2)$. Thus x_1 and x_2 are nonadjacent twins in a vertex critical graph which is impossible.

Thus we may assume that $|A| \geq 3$. If A contained an induced P_3, then G would have an induced $E_2 * (K_1 * B)$. For $K_1 * B$ to be the disjoint union of complete subgraphs and at most one P_3, B must either be E_2 or complete, both of which are impossible. Hence A is a disjoint union of at least two complete subgraphs. The same goes for B.

Assume that A is edgeless. Then, by Lemma 4.44, B must be E_3 or $\overline{P_3}$. Hence $\Delta(G)+1 < |A| + |B| = 6$, giving the contradiction $\Delta(G) \leq 4$.

Since A is the disjoint union of at least two complete subgraphs and contains an edge, it contains $\overline{P_3}$. By Lemma 4.48, B must be either E_3 or the disjoint union of a vertex and a complete subgraph. As above, $B = E_3$ is impossible. In particular B contains $\overline{P_3}$ and using Lemma 4.48 again, we conclude that A is the disjoint union of a vertex and a complete subgraph giving the final contradiction $\omega(G) \geq \omega(A * B) \geq \omega(A) + \omega(B) \geq |A| + |B| - 2 \geq \Delta(G)$. □

Lemma 3.2. *Fix $k \geq 5$. If $G \in \mathcal{C}_k$, then $\overline{G}$ is maximally connected; that is, $\kappa(\overline{G}) = \delta(\overline{G})$.*

Proof. Let $G \in \mathcal{C}_k$ and let S be a cutset in $\overline{G}$ with $|S| = \kappa(\overline{G})$. To get a contradiction, assume that $|S| < \delta(\overline{G}) = |G| - (\Delta(G) + 1)$. Since $\overline{G} - S$ is disconnected, $G - S = A * B$ for some graphs A and B with $1 \leq |A| \leq |B|$. We have $|A| + |B| = \left|\overline{G} - S\right| = |G| - |S| > |G| - (|G| - (\Delta(G) + 1)) = \Delta(G) + 1$. But then Lemma 3.1 gives a contradiction. □

4 List coloring lemmas

In this section we use list-coloring lemmas to forbid a large class of graphs from appearing as subgraphs of mules. In each case, we assume that such a graph $H \lhd G$ appears as an induced subgraph of a mule G. By the minimality of G, we can color $G \setminus H$ with $\Delta - 1$ colors. If H can be colored regardless of which colors are forbidden by its colored neighbors in $G \setminus H$, then we can clearly extend this coloring to all of G. We use the term d_1-choosable to describe such a graph H.

We characterize all graphs $A * B$ with $|A| \geq 2$, $|B| \geq 2$ that are not d_1-choosable. The characterization is somewhat lengthy, so we split it into a number of lemmas. For the case $|A| \geq 4$, $|B| \geq 4$, see Lemma 4.49. When $|A| = 3$, we consider the four cases $A = E_3$ (Lemma 4.44), $A = \overline{P_3}$ (Lemma 4.48), $A = P_3$ (Lemma 4.31), and $A = K_3$ (Lemma 4.34). When $|A| = 2$, we consider the case $A = E_2$ in Lemma 4.27 and the case $A = K_2$ in Lemma 4.52. Finally, in Lemma 4.58, we characterize all triangle-free graphs B such that $K_1 * B$ is not d_1-choosable.

Let G be a graph. A *list assignment* to the vertices of G is a function from $V(G)$ to the finite subsets of $\mathbb{N}$. A list assignment L to G is *good* if G has a coloring c where $c(v) \in L(v)$ for each $v \in V(G)$. It is *bad* otherwise. We call the collection of all colors that appear in L, the *pot* of L. That is $Pot(L) := \bigcup_{v \in V(G)} L(v)$. For a subgraph H of G we write $Pot_H(L) := \bigcup_{v \in V(H)} L(v)$. For $S \subseteq Pot(L)$, let G_S be the graph $G[\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}]$. We also write G_c for $G_{\{c\}}$. We let $\mathcal{B}(L)$ be the bipartite graph that has parts $V(G)$ and $Pot(L)$ and an edge from $v \in V(G)$ to $c \in Pot(L)$ iff $c \in L(v)$. For $f \colon V(G) \to \mathbb{N}$, an f-assignment on G is an assignment L of lists to the vertices of G such that $|L(v)| = f(v)$ for each $v \in V(G)$. We say that G is *f-choosable* if every f-assignment on G is good.

4.1 Shrinking the pot

In this section we prove a lemma about bad list assignments with minimum pot size. Some form of this lemma has appeared independently in at least two places we know of—Kierstead [8] and Reed and Sudakov [17]. We will use this lemma repeatedly in the arguments that follow.

Given a graph G and $f \colon V(G) \to \mathbb{N}$, we have a partial order on the f-assignments to G given by $L < L'$ iff $|Pot(L)| < |Pot(L')|$. When we talk of *minimal* f-assignments, we mean minimal with respect to this partial order.

Lemma 4.1. *Let G be a graph and $f \colon V(G) \to \mathbb{N}$. Assume G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq Pot(L)$, any coloring of G_S from L uses some color not in S.*

Proof. Suppose not and let $\emptyset \neq S \subseteq Pot(L)$ be such that G_S has a coloring ϕ from L using only colors in S. For $v \in V(G)$, let $h(v)$ be the smallest element of $Pot(L) - L(v)$ (this is well defined by assumption). Pick some $c \in S$ and construct a new list assignment L' as follows.

$$L'(v) = \begin{cases} L(v) & \text{if } v \in V(G) - V(G_S) \\ L(v) & \text{if } v \in V(G_S) \text{ and } c \notin L(v) \\ (L(v) - \{c\}) \cup \{h(v)\} & \text{if } v \in V(G_S) \text{ and } c \in L(v) \end{cases}$$

Note that L' is an f-assignment and $Pot(L') = Pot(L) - \{c\}$. Thus, by minimality of L, we can properly color G from L'. In particular, we have a coloring of $V(G) - V(G_S)$ from L using no color from S. We can complete this to a coloring of G from L using ϕ. This contradicts the fact that L is bad. □

Definition 5. A bipartite graph with parts A and B has *positive surplus* (with respect to A) if $|N(X)| > |X|$ for all $\emptyset \neq X \subseteq A$.

Lemma 4.2. *Let G be a graph and $f \colon V(G) \to \mathbb{N}$. Assume G is not f-choosable and let L be a minimal bad f-assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then $\mathcal{B}(L)$ has positive surplus (with respect to $Pot(L)$).*

Proof. Suppose not and choose $\emptyset \neq X \subseteq Pot(L)$ such that $|N(X)| \leq |X|$ minimizing $|X|$. If $|X| = 1$, then G_X can be colored from X contradicting Lemma 4.1. Hence $|X| \geq 2$.

By minimality of $|X|$, for any $Y \subset X$, $|N(Y)| \geq |Y| + 1$. Hence, for any $x \in X$, we have $|N(X)| \geq |N(X - \{x\})| \geq |X - \{x\}| + 1 = |X|$. Thus, by Hall's Theorem, we have a matching of X into $N(X)$, but $|N(X)| \leq |X|$ so this gives a coloring of G_X from X contradicting Lemma 4.1. □

Our approach to coloring a graph (particularly a join) will often be to consider nonadjacent vertices u and v and show that their lists contain a common color. By the pigeonhole principle, this follows immediately when $|L(u)|+|L(v)| > |Pot(L)|$. We will use the following lemma frequently throughout the remainder of this paper.

Small Pot Lemma. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If G is not f-choosable, then G has a minimal bad f-assignment L such that $|Pot(L)| < |G|$.*

Proof. Suppose not and let L be a minimal bad f-assignment. For each $v \in V(G)$ we have $|L(v)| = f(v) < |G| \leq |Pot(L)|$ and hence $L(v) \neq Pot(L)$. Thus by Lemma 4.2 we have the contradiction $|G| \geq |N(Pot(L))| > |Pot(L)|$. □

4.2 Degree choosability

Definition 6. Let G be a graph and $r \in \mathbb{Z}$. Then G is *d_r-choosable* if G is f-choosable where $f(v) = d(v) - r$.

Note that a vertex critical graph with $\chi = \Delta + 1 - r$ contains no induced d_r-choosable subgraph. Since we are working to prove the Borodin-Kostochka Conjecture, we will focus on the case $r = 1$ and primarily study d_1-choosable graphs. For $r = 0$, we have the following well known generalization of Brooks' Theorem (see [2], [6], [11], [5] and [7]).

Definition 7. A *Gallai tree* is a graph all of whose blocks are complete graphs or odd cycles.

Classification of d_0-choosable graphs. *For any connected graph G, the following are equivalent.*

- *G is d_0-choosable.*
- *G is not a Gallai tree.*
- *G contains an induced even cycle with at most one chord.*

We give a couple lemmas about d_0-assignments that will be useful in our study of d_1-assignments. The following lemma was used in [11].

Lemma 4.3. *Let L be a bad d_0-assignment on a connected graph G and $x \in V(G)$ a noncutvertex. Then $L(x) \subseteq L(y)$ for each $y \in N(x)$.*

Proof. Suppose otherwise that we have $c \in L(x) - L(y)$ for some $y \in N(x)$. Coloring x with c leaves at worst a d_0-assignment L' on the connected $H := G - x$ where $|L'(y)| > d_H(y)$. But then we can complete the coloring, a contradiction. □

Lemma 4.4. *If L is a bad d_0-assignment on a connected graph G, $|Pot(L)| < |G|$.*

Proof. Suppose that the lemma is false and choose a connected graph G together with a bad d_0-assignment L where $|Pot(L)| \geq |G|$ minimizing $|G|$. Plainly, $|G| \geq 2$. Let $x \in G$ be a noncutvertex (any end block has at least one). By Lemma 4.3, $L(x) \subseteq L(y)$ for each $y \in N(x)$. Thus coloring x decreases the pot by at most one, giving a smaller counterexample. This contradiction completes the proof. □

We also need a few basic lemmas about how d_r-choosability behaves with respect to induced subgraphs.

Lemma 4.5. *Fix $r \geq 0$. Let G be a graph and $H \trianglelefteq G$ a d_r-choosable subgaph. If L is a d_r-assignment on G and $G - H$ is properly colorable from L, then G is properly colorable from L.*

Proof. Color $G - H$ from L. Let L' be the resulting list assignment on H. Since each $v \in V(H)$ must be adjacent to as many vertices as colors in $G - H$ we see that L' is again a d_r-assignment. The lemma follows. □

Lemma 4.6. *Fix $r \geq 0$. Let G be a graph and $H \trianglelefteq G$ a d_r-choosable subgaph. If there exists an ordering $v_1, \dots, v_t$ of the vertices of $G - H$ such that v_i has degree at least $r + 1$ in $G[V(H) \cup \bigcup_{1 \leq j \leq i-1} v_j]$ for each i, then G is d_r-choosable.*

Proof. Let L be a d_r-assignment on G. Go through $G - H$ in order $v_t, \dots, v_1$ coloring v_i with the smallest available color in $L(v_i)$. Since when we go to color v_i, it has at least $r + 1$ uncolored neighbors we succeed in coloring $G - H$. Now the lemma follows from Lemma 4.5. □

We will also use the following immediate consequence of the pigeonhole principle.

Lemma 4.7. *If $S_1, \dots, S_m$ are nonempty subsets of a finite set T and $\sum_{i \geq 1} |S_i| > (m-1)|T|$, then $\bigcap_{i \geq 1} S_i \neq \emptyset$.*

4.3 Handling joins

The main result of this section is Lemma 4.14, which plays a key role in our classification of bad graphs $A * B$. Specifically, Lemma 4.14 is essential to the proof of Lemma 4.23, which considers the case when $|A| \geq 4$ and B is arbitrary.

Lemma 4.8. *Fix $r \geq 0$. Let A be a graph with $|A| \geq r + 1$ and B a nonempty graph. If $A * B$ is d_r-choosable, then $A * C$ is d_r-choosable for any graph C with $B \trianglelefteq C$.*

Proof. Assume $A * B$ is d_r-choosable and let C be a graph with $B \trianglelefteq C$. Put $H = C - B$. For each $v \in V(H)$, $|L(v)| \geq d(v) - r \geq d_H(v) + r + 1 - r = d_H(v) + 1$. Thus we may color H from its lists. By Lemma 4.5, we can complete the coloring to all of $A * C$. □

Lemma 4.9. *Fix $r \geq 0$. Let A be a graph with $|A| \geq r$ and B a nonempty graph. If $A * B$ is d_r-choosable, then $A * C$ is d_r-choosable for any connected graph C with $B \trianglelefteq C$.*

Proof. Assume $A * B$ is d_r-choosable and let C be a connected graph with $B \trianglelefteq C$. Put $H = C - B$. For each $v \in H$, $|L(v)| \geq d(v) - r \geq d_H(v) + r - r = d_H(v)$. Since C is connected, each component of H has a vertex v that hits a vertex in B and hence has $|L(v)| \geq d_H(v) + 1$. Thus we may color H from its lists. By Lemma 4.5, we can complete the coloring to all of $A * C$. □

Lemma 4.10. *Fix $r \geq 0$. Let G be a d_{r-1} choosable graph with at least $2r+2$ vertices. Then $E_2 * G$ is d_r-choosable.*

Proof. Let x, y be the vertices in the E_2. Suppose $E_2 * G$ is not d_r-choosable. Then by the Small Pot Lemma, we have a d_r-assignment L with $|Pot(L)| < 2 + |G|$. Now $|L(x)| + |L(y)| \geq d(x) + d(y) - 2r \geq 2\,|G| - 2r \geq 2 + |G| > |Pot(L)|$, since $|G| \geq 2r + 2$. Thus we can use a single common color on x and y, leaving a d_{r-1}-assignment on G. We may now complete the coloring, giving a contradiction. □

Since every graph is d_{-1}-choosable we get immediately.

Corollary 4.11. *For $r \geq 0$, both E_2^{r+2} and $E_2^{r+1} * K_2$ are d_r-choosable.*

Lemma 4.12. *Fix $r \geq 0$. Let A be a graph with $|A| \geq 3r + 2$ and B an arbitrary graph. If $A * B$ is not d_r-choosable, then $\omega(B) \geq |B| - 2r$.*

Proof. Suppose $G := A * B$ is not d_r-choosable and let L be a minimal bad d_r-assignment. Then, by the Small Pot Lemma, $|Pot(L)| \leq |G| - 1$. Let $g \colon S \to Pot_S(L)$ be a partial coloring of B from L maximizing $|S| - |im(g)|$ and then minimizing $|S|$. Color S using g and let L' be the resulting list assignment.

Put $H := G - S$ and $C := B - S$. First suppose that $|S| - |im(g)| \geq r + 1$. For each $v \in C$ we have $|L'(v)| \geq d_C(v) - r + 3r + 2 > d_C(v)$, so we can complete g to C. This leaves each $v \in V(A)$ with a list of size at least $d_A(v) - r + |S| - |im(g)| > d_A(v)$. Hence, we can complete the coloring to all of G. Thus L is not bad after all, giving a contradiction.

So instead we assume that $|S| - |im(g)| \leq r$. By the minimality condition on $|S|$ we see that g has no singleton color classes. In particular, $|S| \geq 2\,|im(g)|$. By combining this inequality with $|S| - |im(g)| \leq r$, we get $|S| \leq 2r$. Since $|C| = |B| - |S| \geq |B| - 2r$, the conclusion will follow if we can show that C is complete.

By definition, $|Pot(L')| = |Pot(L)| - |im(g)|$. By the maximality condition on g, every pair of nonadjacent vertices in C must have disjoint lists under L' (otherwise we could use a common color on nonadjacent vertices in C and increase $|S| - |im(g)|$). Let I be a maximal independent set in C. To reach a contradiction, we assume that $|I| \geq 2$. Then for all the elements of I to have disjoint lists, we must have

$$\begin{aligned}
\sum_{v \in I} |L'(v)| &\le |Pot(L')| \\
\sum_{v \in I} (d_H(v) - r) &\le |Pot(L')| \\
\sum_{v \in I} (|A| + d_C(v) - r) &\le |Pot(L')| \\
(|A| - r)\,|I| + \sum_{v \in I} d_C(v) &\le |Pot(L')| \\
(|A| - r)\,|I| + |C| - |I| &\le |Pot(L')| \\
(|A| - r - 1)\,|I| + |B| - |S| &\le |A| + |B| - 1 - |im(g)| \\
(|A| - r - 1)\,|I| &\le |A| - 1 + |S| - |im(g)| \\
2(|A| - r - 1) &\le |A| - 1 + |S| - |im(g)| \\
|A| - 2r - 1 &\le |S| - |im(g)| \\
r + 1 &\le |S| - |im(g)|\,.
\end{aligned}$$

This final inequality contradicts our assumption that $|S| - |im(g)| \le r$. Hence $|I| \le 1$; that is, C is complete. □

Lemma 4.13. *Fix $r \ge 1$. Let A be a connected graph and B an arbitrary graph such that $A * B$ is not d_r-choosable. Let L be a minimal bad d_r-assignment on $A * B$. If B is colorable from L using at most $|B| - r$ colors, then $|Pot(L)| \le |A| + |B| - 2$.*

Proof. By the Small Pot Lemma, $|Pot(L)| \le |A| + |B| - 1$, so to get a contradiction suppose that $|Pot(L)| = |A| + |B| - 1$ and that B is colorable from L using at most $|B| - r$ colors. If $|Pot_A(L)| \ge |Pot(L)| + 1 - r$, then coloring B with at most $|B| - r$ colors leaves at worst a d_0-assignment L' on A with $|Pot(L')| \ge |A|$. Hence the coloring can be completed to A by Lemma 4.4, a contradiction.

Thus we may assume that $|Pot_A(L)| \le |Pot(L)| - r$. Put $S := Pot(L) - Pot_A(L)$. Let π be a coloring of B from L using at most $|B| - r$ colors, say π uses colors C. Then $|C| = |B| - r$ and $S \cap C = \emptyset$ for otherwise coloring B leaves at worst a d_{-1}-assignment on A. Also, $\pi^{-1}(c) \not\subseteq V(G_S)$ for any $c \in C$ since otherwise we could recolor $\pi^{-1}(c)$ with colors from S to get at worst a d_{-1}-assignment on A. In particular, $|G_S| \le \sum_{c \in C} (|\pi^{-1}(c)| - 1) = |B| - |C| = r \le |S|$. But this inequality contradicts Lemma 4.2. □

We now use Lemma 4.13 to strengthen Lemma 4.12.

Lemma 4.14. *Fix $r \ge 1$. Let A be a connected graph with $|A| \ge 3r + 1$ and B an arbitrary graph. If $A * B$ is not d_r-choosable, then $\omega(B) \ge |B| - 2r$.*

Proof. Suppose $G := A * B$ is not d_r-choosable and let L be a minimal bad d_r-assignment. Then, by the Small Pot Lemma, $|Pot(L)| \le |G| - 1$. Let $g \colon S \to Pot_S(L)$ be a partial coloring of B from L maximizing $|S| - |im(g)|$ and then minimizing $|S|$. Color S using g and let L' be the resulting list assignment.

Put $C := B - S$. Running through the argument in Lemma 4.12 with $3r + 1$ in place of $3r + 2$ shows that we must have $|S| - |im(g)| = r$. But then completing g to C gives a coloring of B from L using at most $|B| - r$ colors. Thus, by Lemma 4.13, $|Pot(L)| \leq |G| - 2$. Now running through the argument in Lemma 4.12 again completes the proof. □

4.4 The r = 1 case

4.4.1 Some preliminary tools

The Small Pot Lemma says that if $A * B$ is not d_1-choosable, then $A * B$ has a bad d_1-assignment L such that $|Pot(L)| \leq |A| + |B| - 1$. In this section, we study conditions under which $|Pot(L)| \leq |A| + |B| - 2$. We also prove a key lemma for coloring graphs of the form $K_1 * B$. In the following section, our results here help us to find nonadjacent vertices with a common color.

Lemma 4.15. *Let A be a graph with $|A| \geq 2$, B an arbitrary graph and L a d_1-assignment on $A * B$. If B has an independent set I such that $(|A| - 1)|I| + |E_B(I)| > |Pot(L)|$, then B can be colored from L using at most $|B| - 1$ colors.*

Proof. Suppose that B has an independent set I such that $(|A| - 1)|I| + |E(I)| > |Pot(L)|$. Now

$$\sum_{v \in I} |L(v)| = \sum_{v \in I} (d(v) - 1) = (|A| - 1)|I| + \sum_{v \in I} d_B(v) = (|A| - 1)|I| + |E_B(I)| > |Pot(L)|.$$

Hence we have distinct $x, y \in I$ with a common color c in their lists. So we color x and y with c. Since $|A| \geq 2$, this leaves at worst a d_{-1}-assignment on the rest of B. Completing the coloring to the rest of B gives the desired coloring of B from L using at most $|B| - 1$ colors. □

Lemma 4.16. *Let G be a graph and I a maximal independent set in G. Then $|E(I)| \geq |G| - |I|$. If I is maximum and $|E(I)| = |G| - |I|$, then G is the disjoint union of $|I|$ complete graphs.*

Proof. Each vertex in $G - I$ is adjacent to at least one vertex in I. Hence $|E(I)| \geq |G| - |I|$.

Now assume I is maximum and $|E(I)| = |G| - |I|$. Then $N(x) \cap N(y) = \emptyset$ for every distinct pair $x, y \in I$. Also, $N(x)$ must be a clique for each $x \in I$, since otherwise we could swap x out for a pair of nonadjacent neighbors and get a larger independent set. Since we can swap x with any of its neighbors to get another maximum independent set, we see that G has components $\{G[\{v\} \cup N(v)] \mid v \in I\}$. □

Lemma 4.17. *Let A be a connected graph with $|A| \geq 4$ and B an incomplete graph. If $A * B$ is not d_1-choosable, then $A * B$ has a minimal bad d_1-assignment L such that $|Pot(L)| \leq |A| + |B| - 2$.*

Proof. Suppose $A * B$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $A * B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |A| + |B| - 1$. Let I be a maximum independent set in B. Since B is incomplete, $|I| = \alpha(B) \geq 2$. By Lemma 4.16, $|E_B(I)| \geq |B| - |I| = |B| - \alpha(B)$. As $|A| \geq 4$ we have $(|A|-1)|I| + |E_B(I)| \geq (|A|-1)\alpha(B) + |B| - \alpha(B) \geq (|A|-2)\alpha(B) + |B| \geq 2|A| - 4 + |B| > |A| + |B| - 1 \geq |Pot(L)|$. Hence by Lemma 4.15, B can be colored from L using at most $|B|-1$ colors. But then we are done by Lemma 4.13. □

Lemma 4.18. *Let A be a connected graph with $|A| = 3$ and B a graph that is not the disjoint union of at most two complete subgraphs. If $A * B$ is not d_1-choosable, then $A * B$ has a minimal bad d_1-assignment L such that $|Pot(L)| \leq |B| + 1$.*

Proof. Suppose $A * B$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $A * B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |B| + 2$.

Let I be a maximum independent set in B. Since B is not the disjoint union of at most two complete subgraphs, Lemma 4.16 implies that either $|E(I)| > |B| - |I|$ or $|I| \geq 3$. In the first case, $2|I| + |E(I)| > 2|I| + |B| - |I| \geq 2 + |B| \geq |Pot(L)|$. In the second case, $2|I| + |E(I)| \geq 2|I| + |B| - |I| \geq 3 + |B| > |Pot(L)|$.

Thus by Lemma 4.15, B can be colored from L using at most $|B| - 1$ colors. But then we are done by Lemma 4.13. □

Lemma 4.19. *Let B be a graph containing an induced claw, C_4, K_4^-, P_5, bull, or $2P_3$. If $K_2 * B$ is not d_1-choosable, then $K_2 * B$ has a minimal bad d_1-assignment L such that $|Pot(L)| \leq |B|$.*

Proof. Suppose $K_2 * B$ is not d_1-choosable and let L be a minimal bad d_1-assignment on $K_2 * B$. Then, by the Small Pot Lemma, $|Pot(L)| \leq |B| + 1$.

Let H be an induced claw, C_4, K_4^-, P_5, bull or $2P_3$ in B and M a maximum independent set in H. Expand M to a maximal independent set I in B. We can easily verify that in each case $|E_H(M)| \geq |H| - |M| + 2$, which implies that $|E_B(I)| \geq |B| - |I| + 2$. Hence we have $(|K_2| - 1)|I| + |E_B(I)| \geq (|K_2| - 2)|I| + |B| + 2 = |B| + 2 > |Pot(L)|$. Now by Lemma 4.15, B can be colored from L using at most $|B| - 1$ colors. But then we are done by Lemma 4.13. □

In the case that $A = K_1$, we might not be able to finish an arbitrary precoloring of B from L to all of B as we did above. However, if there is a precoloring that has our desired properties, then there is a coloring of B from the lists maintaining these properties. The following lemma makes this precise.

Lemma 4.20. *Let A and B be graphs such that $G := A * B$ is d_0-choosable, but is not d_1-choosable; let L be a bad d_1-assignment on G. Then*

1. *for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$;*
2. *for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds*
   - *(a) $L(x_1) \cap L(y_1) = \emptyset$;*
   - *(b) $L(x_2) \cap L(y_2) = \emptyset$;*
   - *(c) $|L(x_1) \cap L(y_1)| = 1$ and $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$.*

Proof. For both (1) and (2) we prove the contrapositive.

(1) Suppose that B has an independent set I of size 3 such that there exists a color c that appears in the list of each vertex in I; let $I = \{v_1, v_2, v_3\}$. Since G is d_0-choosable, G has an L-coloring. We will modify this coloring to get an L-coloring that uses c on at least three vertices.

For each v_i in I, if v_i does not have a neighbor with color c, we recolor v_i with c. If c now appears three or more times in our current coloring, then we are done. Assume that c appears on either a single vertex w_1 or on two vertices w_1 and w_2.

If both w_1 and w_2 have two neighbors in I, then we uncolor w_1 and w_2 and use color c on all vertices of I. Otherwise, there exists a single vertex, say w_1, with at least two neighbors in I for which w_1 is their only neighbor with color c_1. Uncolor w_1 and now use color c on all of its neighbors in I_1 that no longer have a neighbor with color c_1. Since each uncolored vertex has at least two neighbors with color c, we can extend the coloring to all of B. Now since color c is used 3 or more times on B, at most $|G| - 2$ colors are used on G, so we can extend the coloring to A.

(2) Suppose that B has two disjoint independent sets I_1 and I_2 each of size 2 and there exist distinct colors c_1 and c_2 such that (for each $i \in \{1, 2\}$) color c_i appears in the lists of both vertices of I_i. Since G is d_0-choosable, B has an L-coloring. We will show that B has an L-coloring in which colors c_1 and c_2 each appear twice (or one appears at least three times). We will modify our coloring using recoloring arguments similar to that above, although we may need to recolor repeatedly. (If at any point our coloring of B uses a single color three or more times, then we can stop, since we will be able to extend this coloring to A.)

If c_1 does not appear in our coloring, then we recolor some vertex of I_1 with c_1. Suppose that color c_1 appears only once in our coloring, say on vertex u. Either we can recolor some vertex in I_1 with c_1 or else both vertices in I_1 are adjacent to u. In this case, we uncolor u and use c_1 on both vertices of I_1. Now we have some color available for u. Thus, we may assume that our coloring uses c_1 on exactly two vertices. If neither of these vertices with c_1 are in I_2, then we can use the same recoloring trick for color c_2. Neither vertex with c_1 will get recolored, so afterwards both colors c_1 and c_2 will appear on two vertices (and we'll be able to extend the coloring to A).

Suppose instead that both vertices with color c_1 are in I_2. If neither vertex in I_2 is adjacent to a vertex where color c_2 is used, then we can recolor both of them with c_2. Next we can again apply the recoloring trick for color c_1. Since the vertices in I_2 with color c_2 will not get recolored, this will yield the desired coloring that uses each of c_1 and c_2 twice. So suppose that c_1 is used on both vertices in I_2 and c_2 is used on a vertex adjacent to at least one vertex in I_2. Since we may assume that c_2 appears on only one vertex, when we use the recoloring trick for c_2, we will color at least one vertex of I_2 with c_2. Thus, we may assume (up to symmetry of I_1 and I_2) that color c_1 appears on two vertices and that exactly one of them is in I_2; we may also assume that color c_2 appears on exactly one vertex.

We will show that after applying the recoloring trick at most three times we will get a coloring of B that uses c_1 on two vertices and uses c_2 on two vertices. We call a vertex $v \in I_i$ *miscolored* if it is colored with color c_{3-i}. We will see that each time we apply the recoloring trick, either we increase the total number of vertices colored with c_1 and c_2 or else we decrease the number of miscolored vertices. Since we begin with at most two miscolored vertices, after applying the recoloring trick at most three times, our coloring will use colors

c_1 and c_2 each twice (and we will be done).

Assume that c_1 appears on two vertices and exactly one of them is miscolored; assume that c_2 appears on exactly one vertex, which may or may not be miscolored. When we apply the recoloring trick for c_2, we increase the number of vertices using c_2. Thus, we are done unless we decrease the number of vertices using c_1. Since we only remove color c_1 from vertices in I_2, we conclude that we've reduced the number of miscolored vertices. We now apply the recoloring trick for c_1. Again, we are done unless we've recolored a miscolored vertex. So assume that we did. Since we have no remaining miscolored vertices, when we now apply the recoloring trick for c_2, we get a coloring that uses each of c_1 and c_2 twice. Thus, we can extend the coloring of B to A. □

A simple variation of the (1) case in the above together with Lemma 4.13 gives the following pot-shrinking lemma for $K_1 * H$.

Lemma 4.21. *Let H be a d_0-choosable graph such that $G := K_1 * H$ is not d_1-choosable and L a minimal bad d_1-assignment on G. If some nonadjacent pair in H have intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

Lemma 4.22. *Let A be a connected graph, let $G = A * B$, and suppose that either B is d_0-choosable or $|A| \geq 2$. (1) Let L be a d_1-assignment to G. If B contains disjoint independent sets I_1 and I_2 such that $\sum_{v \in I_1}(d(v) - 1) \geq |Pot(L)| + 1$ and $\sum_{v \in I_2}(d(v) - 1) \geq |Pot(L)| + 2$, then $A * B$ has an L-coloring. (2) In particular, if B contains disjoint independent sets I_1 and I_2 such that $\sum_{v \in I_1}(d(v) - 1) \geq |G| - 1$ and $\sum_{v \in I_2}(d(v) - 1) \geq |G|$, then $A * B$ is d_1-choosable.*

Proof. Let L be a bad d_1-list assignment. We prove (1) and (2) simultaneously. By the Small Pot Lemma, $|Pot(L)| < |G|$. Thus, since $\sum_{v \in I_2}(d(v) - 1) > |Pot(L)|$, we see that some color α appears on nonadjacent vertices in I_2. Either B is d_0-choosable or $|A| \geq 2$, so using either Lemma 4.21 or Lemma 4.13, we get that $|Pot(L)| = |G| - 2$, so $|G| - 1 \geq |Pot(L)| + 1$.

Since $\sum_{v \in I_1}(d(v) - 1) \geq |Pot(L)| + 1$, we see that two vertices of I_1 have a common color β. If β appears 3 times in I_2, then we are done by Lemma 4.20. Otherwise, we use β on the vertices of I_1 where it appears. After deleting β from the lists of I_2, we can find a common color on two vertices of I_2. Again we are done, by Lemma 4.20. □

4.4.2 A classification

In this section we classify the d_1-choosable graphs of the form $A * B$ where $|A| \geq 2$ and $|B| \geq 2$. When $|A| \geq 4$ and A is connected (or similarly for B), the characterizations follows from Lemma 4.25 and Corollary 4.30. The remainder of the section considers the case when each of A and B is small and/or disconnected.

Definition 8. A graph G is *almost complete* if $\omega(G) \geq |G| - 1$.

Lemma 4.23. *Let A be a connected graph with $|A| \geq 4$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is $E_3 * K_{|B|-3}$ or almost complete.*

Proof. Suppose $A * B$ is not d_1-choosable and B is neither $E_3 * K_{|B|-3}$ nor almost complete. Then, by Lemma 4.14, we have $\omega(B) = |B| - 2$.

Let L be a minimal bad d_1-assignment on $A * B$. Then, by Lemma 4.17, $|Pot(L)| \leq |A| + |B| - 2$. Choose $x_1, x_2 \in V(B)$ so that $B - \{x_1, x_2\}$ is complete. Since B is not $E_3 * K_{|B|-3}$ we have $x_1', x_2' \in V(B)$ such that $\{x_1, x_1'\}$ and $\{x_2, x_2'\}$ are disjoint pairs of nonadjacent vertices. We have $|L(x_i)| + |L(x_i')| \geq d(x_i) + d(x_i') - 2 \geq 2|A| + d_B(x_i) + |B| - 5$.

First suppose $d_B(x_i) > 0$ for some $i \in \{1, 2\}$. Without loss of generality, suppose $i = 1$. Then $|L(x_1)| + |L(x_1')| \geq |Pot(L)| + 2$ and $|L(x_2)| + |L(x_2')| \geq |Pot(L)| + 1$. Hence we have different colors c_1, c_2 such that $c_1 \in L(x_1) \cap L(x_1')$ and $c_2 \in L(x_2) \cap L(x_2')$. Coloring the pairs with these colors leaves a list assignment which is easily completable to all of $A * B$.

Hence we must have $d_B(x_1) = d_B(x_2) = 0$. But then $|L(x_i)| + |L(x_i')| \geq |Pot(L)| + 1$ for each $i \in \{1, 2\}$ and thus both $L(x_1) \cap L(x_1')$ and $L(x_2) \cap L(x_2')$ are nonempty. If they have different colors in common, we can finish as above. If they have the same color c in common, then coloring x_1, x_2 and x_1' with c leaves a list assignment which is easily completable to all of $A * B$. □

Lemma 4.24. *Let A be a connected graph with $|A| \geq 6$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is almost complete.*

Proof. Suppose $A * B$ is not d_1-choosable. By Lemma 4.23, B is $E_3 * K_{|B|-3}$ or almost complete. Suppose that B is $E_3 * K_{|B|-3}$ and let x_1, x_2, x_3 be the vertices in the E_3.

Let L be a minimal bad d_1-assignment on $A * B$. Then, by Lemma 4.17, $|Pot(L)| \leq |A| + |B| - 2$. We have $\sum_{i=1}^{3} |L(x_i)| \geq \sum_{i=1}^{3} (d(x_i) - 1) = 3(|A| + |B| - 4)$. Since $|B| \geq 3$ we have $|A| + |B| \geq 9$ and hence $3(|A| + |B| - 4) > 2(|A| + |B| - 2) \geq 2|Pot(L)|$. Thus, by Lemma 4.7, we have $c \in \bigcap_{i=1}^{3} L(x_i)$. Coloring x_1, x_2 and x_3 with c leaves a list assignment which is easily completable to the rest of $A * B$. This is a contradiction. Hence B is almost complete. □

When A is incomplete we can do much better.

Lemma 4.25. *Let A be a connected incomplete graph with $|A| \geq 4$ and B an arbitrary graph. If $A * B$ is not d_1-choosable, then B is complete.*

Proof. By Lemma 4.8 it will suffice to show that $A * E_2$ is d_1-choosable. Suppose not and let L be a minimal bad d_1-assignment on $A * E_2$. Then, by Lemma 4.17, $|Pot(L)| \leq |A|$. Let x_1 and x_2 be the vertices in the E_2. Then $|L(x_1)| + |L(x_2)| \geq d(x_1) + d(x_2) - 2 = 2|A| - 2 \geq |Pot(L)| + 2$. Hence we have different $c_1, c_2 \in L(x_1) \cap L(x_2)$.

First, suppose there exists $y \in V(A)$ such that $\{c_1, c_2\} \not\subseteq L(y)$. Without loss of generality, assume $c_1 \notin L(y)$. Then coloring x_1 and x_2 with c_1 leaves a list assignment L' on A where $|L'(v)| \geq d_A(v)$ for all $v \in V(A)$ and $|L'(y)| > d_A(y)$. Hence the coloring can be completed, a contradiction.

Hence $\{c_1, c_2\} \subseteq L(v)$ for all $v \in V(A)$. If $\alpha(A) \geq 3$, then coloring a maximum independent set all with c_1 leaves an easily completable list assignment. Also, if A contains two disjoint pairs of nonadjacent vertices, by coloring one with c_1 and one with c_2 we get another easily completable list assignment. Hence A is almost complete.

Let $z \in V(A)$ such that $A - z$ is complete. Since A is incomplete, we have $w \in V(A - z)$ nonadjacent to z. Also, as A is connected we have $w' \in V(A - z)$ adjacent to z. Color

x_1 and x_2 with c_1 and w and z with c_2 to get a list assignment L' on $D := A - \{w, z\}$ where $|L'(v)| \geq d_D(v)$ for all $v \in V(D)$ and $|L'(w')| > d_D(w')$. Hence the coloring can be completed, a contradiction. □

Lemma 4.26. *$E_2 * 2P_3$ is d_1-choosable.*

Proof. Suppose otherwise. Let the E_2 have vertices x_1 and x_2 and the two P_3's have vertices y_1, y_2, y_3 and y_4, y_5, y_6. By the Small Pot Lemma, we have a minimal bad d_1-assignment on $E_2 * 2P_3$ with $|Pot(L)| \leq 7$. Since $|L(x_1)| + |L(x_2)| = 10 \geq |Pot(L)| + 3$, we have three different colors $c_1, c_2, c_3 \in L(x_1) \cap L(x_2)$. Coloring both x_1 and x_2 with any c_i leaves at worst a d_0-assignment on the $2P_3$. If $c_i \notin L(y_1) \cap L(y_2) \cap L(y_3)$ and $c_i \notin L(y_4) \cap L(y_5) \cap L(y_6)$ for some i, then we can complete the coloring. Thus, without loss of generality, we have $\{c_1, c_2\} \subseteq L(y_1) \cap L(y_2) \cap L(y_3)$ and $c_3 \in L(y_4) \cap L(y_5) \cap L(y_6)$. Color y_1 and y_3 with c_1 and y_4 and y_6 with c_3. Then we can easily complete the coloring on the rest of the $2P_3$. We have used at most 4 colors on the $2P_3$ and hence we can complete the coloring. □

At this point we have enough information to completely classify the d_1-choosable graphs of the form $E_2 * B$.

Lemma 4.27. *$E_2 * B$ is not d_1-choosable iff B is the disjoint union of complete subgraphs and at most one P_3.*

Proof. Suppose we have B such that $E_2 * B$ is not d_1-choosable. By Lemma 4.26, B has at most one incomplete component. Suppose we have an incomplete component C and let $y_1y_2y_3$ be an induced P_3 in C. If $C \neq P_3$, then $|C| \geq 4$ and Lemma 4.25 gives a contradiction. Hence $C = P_3$.

For the other direction, it is easy to see that for any B such that $E_2 * B$ is not d_1-choosable adding a disjoint complete subgraph to B does not make it d_1-choosable. To see that $E_2 * P_3$ is not d_1-choosable, let x_1, x_2 denote the vertices of the E_2 and let y_1, y_2, y_3 denote in order the vertices of the P_3. Let $L(x_1) = \{a, b\}$, $L(x_2) = \{c, d\}$, $L(y_1) = \{a, c\}$, $L(y_2) = \{a, b, c\}$, and $L(y_3) = \{b, d\}$. It is easy to verify that the graph is not colorable from these lists. This proves the lemma. □

For $t \geq 4$, we know that if $K_t * B$ is not d_1-choosable then B is almost complete; or $t = 4$ and B is E_3 or a claw; or $t = 5$ and B is E_3. The following two lemmas show that this completely characterizes the d_1-choosable graphs of this form.

Lemma 4.28. *Almost complete graphs are not d_1-choosable.*

Proof. Let G be almost complete and $x \in V(G)$ such that $G - x$ is complete. Consider the d_1-assignment L given by $L(v) = [d(v) - 1]$ for each $v \in V(G)$. Now $G - x$ is a complete graph of size $|G| - 1$, but the union of the lists on $G - x$ is only $[|G| - 2]$, so by Hall's theorem, G has no coloring from these lists. □

Lemma 4.29. *$K_t * E_3$ is d_1-choosable iff $t \geq 6$.*

Proof. That if $t \geq 6$, then $K_t * E_3$ is d_1-choosable follows from Lemma 4.24. For the other direction it is enough to show that $K_5 * E_3$ is not d_1-choosable. Figure 5 shows a bad d_1-assignment on $K_5 * E_3$. □

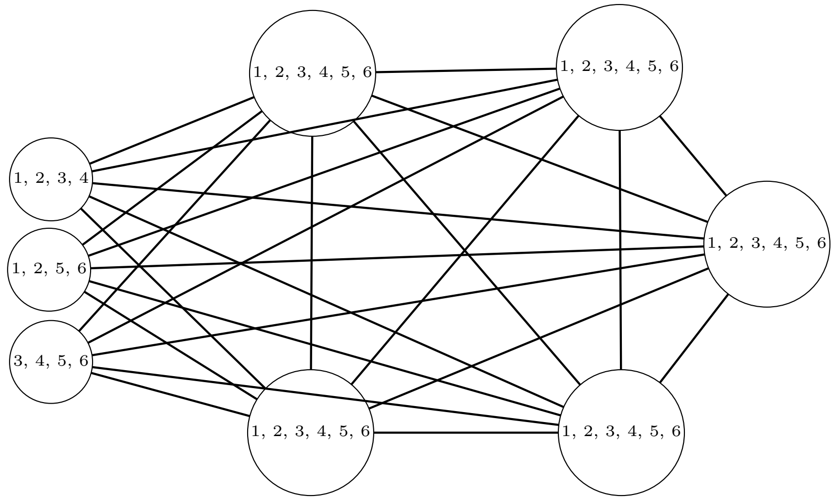


Figure 5: A bad d_1-assignment on $K_5 * E_3$.

Corollary 4.30. *For $t \geq 4$, $K_t * B$ is not d_1-choosable iff B is almost complete; or $t = 4$ and B is E_3 or a claw; or $t = 5$ and B is E_3.*

Lemma 4.31. *$P_3 * B$ is not d_1-choosable iff B is E_2 or complete.*

Proof. Moving the center of P_3 to the other side of the join and applying Lemma 4.27 proves the lemma. □

Lemma 4.32. *$K_3 * P_4$ is d_1-choosable.*

Proof. Suppose otherwise. Denote the vertices of the P_4 as y_1, y_2, y_3, y_4, in order. Note that $|L(y_1)| + |L(y_3)| = 4 + 5 \geq |G| + 1$ and $|L(y_2)| + |L(y_4)| = 5 + 4 \geq |G| + 1$. Now we apply (2) of Lemma 4.22 with $I_1 = \{y_1, y_3\}$ and $I_2 = \{y_2, y_4\}$. □

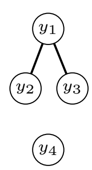


Figure 6: The antipaw.

Lemma 4.33. *$K_3 *$ antipaw is d_1-choosable.*

Proof. Suppose not. We use the labeling of the antipaw given in Figure 6. Since the antipaw is not a disjoint union of at most two complete graphs, Lemma 4.18 gives us a minimal bad d_1-assignment L on $K_3 *$antipaw with $|Pot(L)| \leq 5$. Note that $|L(y_1)| + |L(y_4)| \geq 6$ and $|L(y_2)| + |L(y_3)| \geq 6$. Hence, by Lemma 4.20, $|L(y_1) \cap L(y_4)| = 1$ and $L(y_1) \cap L(y_4) = L(y_2) \cap L(y_3)$. But then we have $c \in L(y_2) \cap L(y_3) \cap L(y_4)$ and after coloring y_2, y_3, and y_4 with c we can complete the coloring, getting a contradiction. □

Lemma 4.34. *$K_3 * B$ is not d_1-choosable iff B is almost complete, $K_t + K_{|B|-t}$, $K_1 + K_t + K_{|B|-t-1}$, $E_3 + K_{|B|-3}$, or $|B| \leq 5$ and $B = E_3 * K_{|B|-3}$.*

Proof. Let $K_3 * B$ be a graph that is not d_1-choosable and let B be none of the specified graphs. Lemma 4.18 gives us a minimal bad d_1-assignment L on $K_3 * B$ with $|Pot(L)| \leq |B| + 1$. Furthermore, the proof of Lemma 4.18 shows that we can color B with at most $|B| - 1$ colors. In particular we have nonadjacent $x, y \in V(B)$ and $c \in L(x) \cap L(y)$. Coloring x and y with c leaves a list assignment L' on $D := B - \{x, y\}$. If $c \in L'(z)$ for some $z \in V(D)$, then $\{x, y, z\}$ is independent and we can color z with c and complete the coloring to get a contradiction. Hence $Pot(L') = Pot(L) - \{c\}$.

Suppose, for a contradiction, that D is not the disjoint union of at most two complete subgraphs. If $\alpha(D) \geq 3$, let J be a maximum independent set in D and set $\gamma := 0$. Otherwise D contains an induced P_3 abc and we let $J \subseteq V(D)$ be a maximal independent set containing $\{a, c\}$ and set $\gamma := 1$. Lemma 4.16 implies that $\sum_{v \in J} d_D(v) \geq |D| - |J| + \gamma$. Since L is bad, we must have

$$\begin{aligned}
\sum_{v \in J} |L'(v)| &\leq |Pot(L')| \\
\sum_{v \in J} |L'(v)| &\leq |B| \\
2|J| + \sum_{v \in J} d_D(v) &\leq |B| \\
2|J| + |D| - |J| + \gamma &\leq |B| \\
|J| + |D| + \gamma &\leq |B| \\
|J| + |B| - 2 + \gamma &\leq |B|.
\end{aligned}$$

Hence $|J| \leq 2 - \gamma$, a contradiction. Therefore D is indeed the disjoint union of at most two complete subgraphs. (Additionally, if D is not complete then $v \in V(D)$ is not adjacent to both x and y since then we would get the same contradictory degree sum as in the case when $\gamma = 1$.) We now consider the case that D is a complete graph and the case that D is the disjoint union of two complete graphs.

First, suppose D is a complete graph. Plainly, $|D| \geq 2$. Put $X := N(x) \cap V(D)$ and $Y := N(y) \cap V(D)$. Suppose $X - Y \neq \emptyset$ and pick $z \in X - Y$. We have $|L(y)| + |L(z)| \geq d(y) + d(z) - 2 = d_B(y) + d_B(z) + 4 \geq 0 + |B| - 2 + 4 = |B| + 2 > |Pot(L)|$. By repeating the argument given above for $B - \{x, y\}$, we see that $B - \{y, z\}$ is also the disjoint union of at most two complete subgraphs. In particular, x is adjacent to all or none of $D - z$. If all, then B is almost complete, if none then B contains an induced P_4 or antipaw, and both possibilities give contradictions by Lemmas 4.32 and 4.33. Hence $X - Y = \emptyset$. Similarly, $Y - X = \emptyset$, so $X = Y$. Since B is not $E_2 + K_{|B|-2}$, $|X| > 0$. If $X = V(D)$, then B is almost complete. If $|V(D) - X| \geq 2$, then pick $w_1, w_2 \in V(D) - X$. Now by considering degrees, we see that $L(x) \cap L(w_1)$ and $L(y) \cap L(w_2)$ are both nonempty. Now we can color x, y, w_1, w_2 using only 2 colors, and then complete the coloring. Hence, we must have $|V(D) - X| = 1$, so let $\{w\} = V(D) - X$. Now x and y are joined to $D - w$ and hence B is $E_3 * K_{|B|-3}$, a contradiction.

Thus D must instead be the disjoint union of two complete subgraphs D_1 and D_2. For each $i \in [2]$, put $X_i := N(x) \cap V(D_i)$ and $Y_i := N(y) \cap V(D_i)$. From our parenthetical remark above, we know that $X_i \cap Y_i = \emptyset$. Suppose we have $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that $L(z_1) \cap L(z_2) \neq \emptyset$. Then, by Lemma 4.20, $L(z_1) \cap L(z_2) = L(x) \cap L(y)$. Since no independent set of size three can have a color in common, the edges z_1x and z_2y or z_1y and z_2x must be present. Using the same argument as for $B - \{x, y\}$, we see that $B - \{z_1, z_2\}$ is the disjoint union of at most two complete subgraphs. So each of x and y is adjacent to all or none of each of $V(D_1 - z_1)$ and $V(D_2 - z_2)$. Thus, by symmetry, we may assume that $V(D_1 - z_1) \subseteq X_1$ and $V(D_2 - z_2) \subseteq Y_2$. If $|D_1| = |D_2| = 1$, then B is the disjoint union of two cliques, a contradiction. So, by symmetry, we may assume that $|D_1| \geq 2$. Pick $w \in V(D_1 - z_1)$. If x is not adjacent to z_1, then xwz_1 is an induced P_3 in B. Since $X_1 \cap Y_1 = \emptyset$, this P_3 together with y either induces a P_4 or an antipaw, contradicting Lemmas 4.32 and 4.33. Hence $X_1 = V(D_1)$. Similarly, if $|D_2| \geq 2$, then $Y_2 = V(D_2)$ and B is the disjoint union of two complete subgraphs, a contradiction. Hence $D_2 = \{z_2\}$. But z_2 must be adjacent to y, so B is again the disjoint union of two cliques, a contradiction.

Thus for every $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ we have $L(z_1) \cap L(z_2) = \emptyset$. Suppose there exist $z_1 \in V(D_1)$ and $z_2 \in V(D_2)$ such that z_1 and z_2 are each adjacent to at least one of x and y. Then $|L(z_1)| + |L(z_2)| \geq d(z_1) + d(z_2) - 2 \geq d_B(z_1) + d_B(z_2) + 4 \geq |B| - 4 + 2 + 4 = |B| + 2 > |Pot(L)|$. Hence $L(z_1) \cap L(z_2) \neq \emptyset$, a contradiction.

Thus, by symmetry, we may assume that there are no edges between D_1 and $\{x, y\}$. Since no vertex in D_2 is adjacent to both x and y, only one of x or y can have neighbors in D_2 lest B contain an induced P_4 contradicting Lemma 4.32. Without loss of generality, we may assume that y has no neighbors in D_2. Pick $w \in D_1$ and $z \in V(D_2)$.

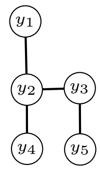
Suppose that $|D_1| \geq 2$, $|D_2| \geq 2$, and there exists $t \in D_2$ such that x and t are nonadjacent. Now choose $u, v \in V(D_1)$ and $w \in V(D_2) \setminus \{t\}$. Now $\{v, w, y\}$ is independent and $|L(v)| + |L(w)| + |L(y|) \geq d(v) + d(w) + d(y) - 3 \geq d_B(v) + d_B(w) + d_B(y) + 6 \geq |B| + 2 > |Pot(L)|$. Hence either $L(v) \cap L(y) \neq \emptyset$ or $L(w) \cap L(y) \neq \emptyset$. Similarly, either $L(u) \cap L(x) \neq \emptyset$ or $L(t) \cap L(x) \neq \emptyset$. Thus, we can color 4 vertices using only 2 colors, and we can complete the coloring. So now either $|D_1| = 1$, $|D_2| = 1$, or $D_2 \subset N(x)$.

If $|D_2| = 1$, then either $B = K_1 + K_2 + K_{|B|-3}$ or else $B = E_3 + K_{|B|-3}$, both of which are forbidden. Similarly, if $|D_1| = 1$ and x is adjacent to all or none of D_2, then $B = K_1 + K_1 + K_{|B|-2}$ or $E_3 + K_{|B|-3}$. Finally, if x is adjacent to some, but not all of D_2, then B contains an antipaw. By Lemma 4.33, this is a contradiction.

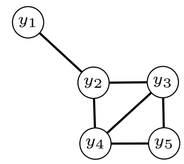
It remains to show that $K_3 * B$ is not d_1-choosable for any of the specified B's. For B almost complete, this follows from Lemma 4.28 and for $E_3 * K_{|B|-3}$, from Lemma 4.30. For all the rest of the options we will give a bad list assignment with lists $[|B| + 1]$ on the K_3. Suppose $K_t + K_{|B|-t}$. On the K_t the lists $[t + 1]$ and on the $K_{|B|-t}$ the lists $[|B| + 1] \smallsetminus [t]$. Then any coloring of $K_3 * B$ from the lists must use three colors on the K_3 and hence at least one of the cliques loses at least two colors leaving it uncolorable. Now suppose $B = K_1 + K_t + K_{|B|-t-1}$. Use the list $\{1, |B| + 1\}$ on the K_1, the lists $[t + 1]$ on the K_t and the lists $[|B + 1|] \smallsetminus [t + 1]$ on the $K_{|B|-t-1}$. This list assignment is clearly bad on $K_3 * B$. Finally suppose $B = E_3 + K_{|B|-3}$. Give the three K_1's the lists $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and the $K_{|B|-3}$ the list $[|B| + 1] \smallsetminus [3]$. Again, this is clearly a bad list assignment on $K_3 * B$. □

Lemma 4.35. *$K_2 * P_5$ is d_1-choosable.*

Proof. Suppose otherwise. By Lemma 4.19, we have a minimal bad d_1-assignment L on $P_5 * K_2$ with $|Pot(L)| \leq 5$. Let y_1, y_2, y_3, y_4, y_5 denote the vertices of the P_5 in order. Now $|L(y_2)| + |L(y_4)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_5)| \geq 7 \geq |Pot(L)| + 2$. So $\{y_2, y_4\}$ and $\{y_1, y_3, y_5\}$ satisfy the hypotheses of Lemma 4.22, giving a contradiction. □



(a) The chair.



(b) The antichair.

Figure 7: Labelings of the chair and the antichair.

Lemma 4.36. *$K_2 *$ chair is d_1-choosable.*

Proof. Suppose otherwise. We use the labeling of the chair given in Figure 7a. Since the chair has an induced claw, Lemma 4.19 gives us a minimal bad d_1-assignment L on $K_2 *$ chair with $|Pot(L)| \leq 5$. Now $|L(y_2)| + |L(y_5)| \geq 6 \geq |Pot(L)| + 1$ and $|L(y_1)| + |L(y_3)| + |L(y_4)| \geq 7 \geq |Pot(L)| + 2$. Then $\{y_2, y_5\}$ and $\{y_1, y_3, y_4\}$ satisfy the hypotheses of Lemma 4.22, giving a contradiction. □

Lemma 4.37. *$K_2 *$ antichair is d_1-choosable.*

Proof. Suppose otherwise. We use the labeling of the antichair given in Figure 7b. Since the antichair has an induced K_4^-, Lemma 4.19 gives us a minimal bad d_1-assignment L on $K_2 *$ antichair with $|Pot(L)| \leq 5$. We have $|L(y_2)| + |L(y_5)| \geq 7$ and hence $|L(y_2) \cap L(y_5)| \geq 2$. But then, by Lemma 4.20, we have the contradiction $|L(y_1)| + |L(y_3)| \leq 5$. □

Lemma 4.38. *$K_2 * C_5$ is d_1-choosable.*

Proof. Suppose otherwise. By the Small Pot Lemma, we have a minimal bad d_1-assignment L on $C_5 * K_2$ with $|Pot(L)| \leq 6$. Let $y_0, y_1, y_2, y_3, y_4, y_0$ denote in order the vertices of the C_5. Then for $0 \leq i < j \leq 4$ with $i - j \not\equiv 1 \pmod 5$ we have $|L(y_i)| + |L(y_j)| \geq d(y_i) + d(y_j) - 2 = 6$.

First suppose $|Pot(L)| \leq 5$. Then each nonadjacent pair has a color in common and by applying Lemma 4.20 multiple times we see that there must exist $c \in \bigcap_{0 \leq i \leq 4} L(y_i)$ and no nonadacent pair can have a color other than c in common. Put $S_i = L(y_i) - \{c\}$ and $T = Pot(L) - \{c\}$. Then we must have $S_0 = T - S_3$, $S_1 = T - S_3 = T - S_4$ and $S_2 = T - S_4$. Hence $S_0 = S_1 = S_2$ contradicting $S_0 \cap S_2 = \emptyset$.

Therefore we must have $|Pot(L)| = 6$. Thus for nonadjacent y_i and y_j, $L(y_i) = Pot(L) - L(y_j)$. We have $L(y_0) = Pot(L) - L(y_3)$, $L(y_1) = Pot(L) - L(y_3) = Pot(L) - L(y_4)$ and $L(y_2) = Pot(L) - L(y_4)$. Hence $L(y_0) = L(y_1) = L(y_2)$. Thus we may color y_0 and y_2 the same and complete this coloring to the rest of B contradicting Lemma 4.13. □

Lemma 4.39. *$K_2 * 2P_3$ is d_1-choosable.*

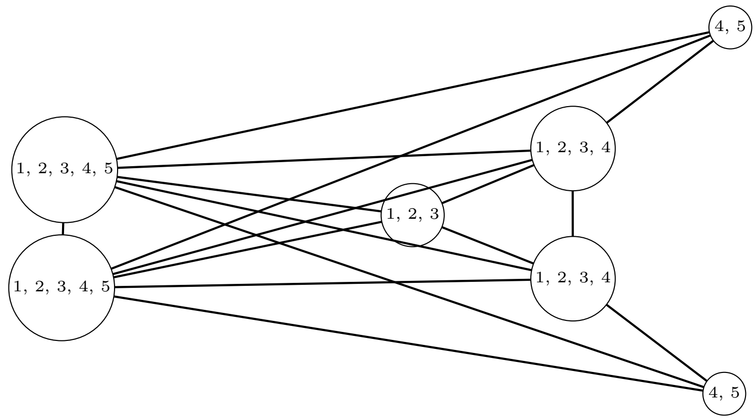


Figure 8: A bad d_1-assignment on bull $* K_2$.

Proof. Suppose otherwise. Let y_1, y_2, y_3 and y_4, y_5, y_6 denote in order the vertices of the two P_3's. Lemma 4.19 gives us a minimal bad d_1-assignment L on $K_2 * 2P_3$ with $|Pot(L)| \leq 6$.

Since $|L(y_1)|+|L(y_3)|+|L(y_4)|+|L(y_6)| = 8 \geq |Pot(L)|+2$, either three of these vertices share a common color, or else two pairs of them share distinct common colors. Thus, if $L(y_2) \cap L(y_5) \neq \emptyset$, then we can color G by Lemma 4.20. Hence $L(y_2) \cap L(y_5) = \emptyset$.

By summing list sizes, we see that some pair among each of $\{y_1, y_3, y_5\}$ and $\{y_2, y_4, y_6\}$ must have a color in common. Since there are no edges between $\{y_1, y_3\}$ and $\{y_4, y_6\}$, if $L(y_1) \cap L(y_3) \neq \emptyset$ and $L(y_4) \cap L(y_6) \neq \emptyset$, then we get a contradiction. By symmetry, we may assume that the other two options are either $L(y_1) \cap L(y_3) \neq \emptyset$ and $L(y_2) \cap L(y_4) \neq \emptyset$ or else $L(y_1) \cap L(y_5) \neq \emptyset$ and $L(y_2) \cap L(y_4) \neq \emptyset$. In the former case, by Lemma 4.20,we must have $L(y_1) \cap L(y_3) \cap L(y_4) \neq \emptyset$, a contradiction. In the latter case, $L(y_1) \cap L(y_5) \neq L(y_2) \cap L(y_4)$ since $L(y_2) \cap L(y_5) = \emptyset$, contradicting Lemma 4.20. □

(a) The anticlaw. (b) The antidiamond.

Figure 9: Labelings of the anticlaw and the antidiamond.

Note that if L is a bad d_1 assignment on $E_3 * B$ where the E_3 is $\{x_1, x_2, x_3\}$, then $L(x_1) \cap L(x_2) \cap L(x_3) = \emptyset$.

Lemma 4.40. *$E_3 *$ anticlaw is d_1-choosable.*

Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad d_1-assignment L on $E_3 *$ anticlaw with $|Pot(L)| \leq 6$. Let the E_3 have vertices x_1, x_2, x_3, and let the anticlaw have vertices y_1, y_2, y_3, y_4, with y_2, y_3, y_4 mutually adjacent. Then $\sum_i |L(x_i)| = 9$ and hence there are three colors c_1, c_2, c_3 such that for each $t \in [3]$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$.

Suppose there exists $i \in \{2, 3, 4\}$, say $i = 2$, such that y_1 and y_i have a common color c. We use c on y_1 and y_2, and let $L'(v) = L(v) - c$ for each uncolored v; note that c must be absent from some x_i, say x_1. Now since $|L'(x_2)| + |L'(x_3)| \geq 4$, we can color x_2 and x_3 such that at least two colors remain available on y_3. Finally, we greedily color y_4, y_3, x_3.

Otherwise, since $|Pot(L)| \leq 6$, we may assume that $L(y_1) = \{a, b\}$ and $L(y_2) = L(y_3) = L(y_4) = \{c, d, e, f\}$. Now we can color x_1, x_2, and x_3 using only two colors, exactly one of which is in $\{a, b\}$. Finally, we greedily color y_1, y_2, y_3, y_4. □

Lemma 4.41. *$E_3 * 2K_2$ is d_1-choosable.*

Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad d_1-assignment L on $E_3 * 2K_2$ with $|Pot(L)| \leq 6$. Let the E_3 have vertices x_1, x_2, x_3, and let the $2K_2$ have vertices y_1 adjacent to y_2 and y_3 adjacent to y_4. Then $\sum_i |L(x_i)| = 9$ and hence there are three colors c_1, c_2, c_3 such that for each $t \in [3]$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$. If all three c_t appear on all four y_i, then we can 2-color the $2K_2$, and extend the coloring to the E_3. So we may assume instead without loss of generality that c_1 appears on x_1 and x_2, but not y_1. Now use c_1 on x_1 and x_2, then color greedily in the order y_3, y_4, x_3, y_2, y_1. □

Lemma 4.42. *$E_3 * E_4$ is d_1-choosable.*

Proof. Suppose otherwise. Let the E_3 have vertices x_1, x_2, x_3 and let the E_4 have vertices y_1, y_2, y_3, y_4. If there exists $c \in \cap_{i=1}^{3} L(x_i)$, then we use c on all x_i and we can finish the coloring, so assume not. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so there exist two y_i, say y_1 and y_2, with a common color c; use c on y_1 and y_2. Now there exists some x_i, say x_3, with $c \notin L(x_i)$. The 4-cycle induced by x_1, x_2, y_3, and y_4 is 2-choosable; then we can extend the coloring to x_3. □

Lemma 4.43. *$E_3 *$ antidiamond is d_1-choosable.*

Proof. Suppose otherwise. The Small Pot Lemma gives us a minimal bad d_1-assignment L on $E_3 *$ antidiamond with $|Pot(L)| \leq 6$. Let the E_3 have vertices x_1, x_2, x_3, and let the antidiamond have vertices y_1, y_2, y_3, y_4, with y_3 adjacent to y_4. We can assume tht $\cap_{i=1}^{3} L(x_i) = \emptyset$ (since otherwise we use a common color on the x_i and then greedily complete the coloring). If y_3 or y_4 has a common color c with y_1 or y_2, then we can use c on those two vertices and proceed as in the case of $E_3 * E_4$, so assume not. Again $\sum_i |L(x_i)| = 9$ and hence there are three colors c_1, c_2, c_3 such that for each $t \in [3]$, $c_t \in L(x_i) \cap L(x_j)$ for some $1 \leq i < j \leq 3$. So assume that c_1 appears on x_1 and x_2, and use it there. If c_1 appears on neither y_1 or y_2, then we greedily color in the order y_3, y_4, x_3, y_1, y_2. Otherwise c_1 appears on neither y_3 or y_4, so we greedily color in the order y_1, y_2, x_3, y_3, y_4. □

Lemma 4.44. *$E_3 * B$ is not d_1-choosable iff $B \in \{K_1, K_2, E_2, E_3, \overline{P_3}, K_3, K_4, K_5\}$.*

Proof. Suppose we have B such that $E_3 * B$ is not d_1-choosable. By Lemma 4.27, B is the disjoint union of complete subgraphs and at most one P_3. If B contained a P_3, then moving its middle vertex to the other side of the join would violate Lemma 4.25. By Lemma 4.42, B has at most three components. By Lemma 4.43, if B has three components, then $B = E_3$. By Lemma 4.41 and Lemma 4.40, if B has two components then $B = E_2$ or $B = \overline{P_3}$. Otherwise B is complete and Lemma 4.29 shows that $|B| \leq 5$. This proves the forward implication.

For the other direction, it is easy to verify that $E_3 * B$ is not d_1-choosable for the listed graphs. The cases $B \in \{K_1, K_2, E_2\}$ are nearly trivial. For $B = E_3$, we are simply recalling that $K_{3,3}$ is not 2-choosable. For $B \in \{K_3, K_4, K_5\}$, see Figure 5. Finally, suppose that $B = \overline{P_3}$. Let x_1, x_2, x_3 denote the vertices of the E_3 and let y_1, y_2, y_3 denote the vertices of the $\overline{P_3}$, where y_2 and y_3 are adjacent. Assign the lists $L(x_1) = \{1, 2\}$, $L(x_2) = \{1, 3\}$, $L(x_3) = \{2, 3\}$, $L(y_1) = \{1, 2\}$, and $L(y_2) = L(y_3) = \{1, 2, 3\}$. To color the $\overline{P_3}$, we clearly use at least two colors, but now some vertex of the E_3 has no remaining colors. □

Lemma 4.45. *$\overline{P_3} * 2K_2$ is d_1-choosable.*

Proof. Let x_1, x_2, x_3 be the vertices of $\overline{P_3}$, with x_2 adjacent to x_3, and let y_1, y_2, y_3, y_4 be the vertices of $2K_2$, with y_1 adjacent to y_2 and y_3 adjacent to y_4. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so x_1 and x_2 have a common color c_1. If c_1 is absent from the list of some y_i, say y_1, then we can use c_1 on x_1 and x_2, then greedily color in the order y_4, y_3, x_3, y_2, y_1. Hence c_1 appears on all y_i. If $|Pot(L)| \leq 5$, then x_1 and x_2 have a second common color c_2. Since c_1 and c_2 must appear on all y_i, we can 2-color the $2K_2$, then greedily color x_1, x_2, and x_3. So we can conclude that $L(x_1) \cap L(x_2) = c_1$ and $L(x_1) \cap L(x_3) = c_1$. Similarly, we can 2-color the $2K_2$ if y_1 and y_3 have any common color other than c_1.

Now we use c_1 on y_2 and y_4, and let $L'(v) = L(v) - c_1$ for all uncolored v. Now $|Pot(L')| = |Pot(L)| - 1 = 5$. Let $S = \{x_1, x_2, x_3, y_1, y_3\}$. To show that we can finish the coloring, we use Hall's Theorem. We only need to consider subsets $T \subset S$ of size 3 or 4. If $|T| = 3$, then either $\{y_1, y_3\} \subset T$, so $|\cup_{v \in T} L'(v)| \geq |L'(y_1)| + |L'(y_3)| \geq 4$, or else T contains x_2 or x_3. Since $|L'(x_2)| = |L'(x_3)| = 3$, we are done. If $|T| = 4$, then either $\{y_1, y_3\} \subset T$ or $\{x_1, x_2\} \subset T$ or $\{x_1, x_3\} \subset T$. In each case $|\cup_{v \in T} L'(v)| \geq 4$. □

Lemma 4.46. *$\overline{P_3} *$ antidiamond is d_1-choosable.*

Proof. Let x_1, x_2, x_3 be the vertices of $\overline{P_3}$, with x_2 adjacent to x_3, and let y_1, y_2, y_3, y_4 be the vertices of the antidiamond, with y_3 adjacent to y_4. By the Small Pot Lemma, $|Pot(L)| \leq 6$, so x_1 and x_2 have a common color c. If c is absent from y_4, then we use c on x_1 and x_2, then greedily color y_1, y_2, x_3, y_3, y_4. Similarly, if c is absent from y_1 and y_2, then we use c on x_1 and x_2, then greedily color y_3, y_4, x_3, y_2, y_1. So c must appear on y_1 (or y_2) and y_3, and we use it there. Let $L'(v) = L(v) - c$ for all uncolored vertices. Now if there exists $c_2 \in L'(y_2) \setminus L'(x_2)$, then we can use c_2 on y_2 and greedily color x_1, y_4, x_3, x_2. The same argument holds if there exists $c_2 \in L'(y_4) \setminus L'(x_2)$. Thus, we must have $(L'(y_2) \cup L'(y_4)) \subseteq L'(x_2)$, so y_2 and y_4 have a common color c_2. We use it on them and greedily color x_1, x_2, x_3. □

Lemma 4.47. *$\overline{P_3} * E_4$ is d_1-choosable.*

Proof. Let x_1, x_2, x_3 be the vertices of $\overline{P_3}$, with x_2 adjacent to x_3, and let y_1, y_2, y_3, y_4 be the vertices of E_4. If three of the y_i's (say y_1, y_2, and y_3) have a common color c, then use c on them, and now greedily color in the order y_4, x_1, x_2, x_3. By the Small Pot Lemma, x_1 and x_2 have a common color c, which we use on them. Now c appears on at most two y_i, say y_1 and y_2, so we can greedily color in the order y_1, y_2, x_3, y_3, y_4. □

Lemma 4.48. *$\overline{P_3} * B$ is not d_1-choosable iff B is E_3, $K_{|B|}$, or $K_1 + K_{|B|-1}$.*

Proof. Since $\overline{P_3}$ contains an E_2, Lemma 4.27 shows that B is the disjoint union of complete subgraphs and at most one P_3. If B contained a P_3, then moving its middle vertex to the other side of the join would violate Lemma 4.25. By Lemma 4.45 at most one component of

B has more than one vertex. If B has more than two components, then Lemma 4.46 shows that B is independent and thus Lemma 4.47 shows that $B = E_3$. If B has two components then it is $K_1 + K_{|B|-1}$. Otherwise B is complete. This proves the forward implication.

The reverse implication is easily checked. For $B = E_3$, see Lemma 4.44. If $B = K_{|B|}$, then G is almost complete. Suppose that $B = K_{|B|-1|}$. Now $\Delta(G) = \omega(G) = |B| + 1$, so G is not d_1-choosable. ☐

Lemma 4.49. *Let A and B be graphs with $|A| \geq 4$ and $|B| \geq 4$. The graph $A * B$ is not d_1-choosable iff $A * B$ is almost complete, $K_5 * E_3$, or $\big(K_1 + K_{|A|-1}\big) * \big(K_1 + K_{|B|-1}\big)$.*

Proof. Suppose A and B are graphs with $|A| \geq |B| \geq 4$ such that $A * B$ is not d_1-choosable and not one of the specified graphs.

First suppose A is connected. If A is complete then by Corollary 4.30, $|A| = 4$ and B is a claw or B is almost complete. But this implies that $G = K_5 * E_3$ or G is almost complete. Hence A is incomplete. Now Lemma 4.25 shows that B is complete. By reversing the roles of A and B in this argument, we get a contradiction; so A is disconnected. The same argument shows that B is also disconnected.

Suppose $\alpha(A) \geq 3$. Then Lemma 4.44 shows that B is K_4 or K_5, both impossible as above. Thus $\alpha(A) = 2$ and hence A is the disjoint union of two complete graphs. The same goes for B. Now Lemma 4.48 shows that $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.

The reverse implication is easily checked. If $A * B$ is almost complete, then clearly it is not d_1-choosable. For $A * B = K_5 * E_3$, see Figure 5. So suppose that $A * B = (K_1 + K_{|A|-1}) * (K_1 + K_{|B|-1})$. Now $\Delta(A * B) = \omega(A * B) = |A| + |B| - 2$, so $A * B$ is not d_1-choosable. ☐

4.4.3 Joins with K_2

Definition 9. The *net* is formed by adding one edge incident to each vertex of K_3. The *bowtie* is formed by identifying one vertex in each of two copies of K_3. The M is formed from the bowtie by adding an edge incident to a vertex of degree 2.

Lemma 4.50. *The graph $K_2 * A$ is d_1-choosable for all $A \in \{2P_3, C_4, C_5, P_5, chair, antichair, K_1 * antipaw, K_1 * P_4, net, M\}$*

Proof. For eight of these ten choices of A, we have already proved that $K_2 * A$ is d_1-choosable. Specifically, we have proved this for $2P_3$ (Lemma 4.39), C_5 (Lemma 4.38), P_5 (Lemma 4.35), chair (Lemma 4.36), antichair (Lemma 4.37), $K_1 *$ antipaw (Lemma 4.33), $K_1 * P_4$ (Lemma 4.32), and C_4 (since $C_4 = E_2^2$, this is the case $r = 1$ in Corollary 4.11). Now we consider the remaining two cases: net and M.

Let $G = K_2 *$ net. Let x_1, x_2 denote the vertices of the K_2, let y_1, y_2, y_3 denote the degree-3 vertices in the net, and let z_1, z_2, z_3, denote the leaves of the net, with z_i adjacent to y_i. We consider three cases. (1) If there exists $c_1 \in \cap_{i=1}^3 L(z_i)$, then we first use c_1 on all three z_i and afterwards color y_1, y_2, y_3, x_1, x_2 greedily. (2) Suppose there exist y_i and z_j, with $i \neq j$, such that there exists $c_1 \in L(y_i) \cap L(z_j)$; by symmetry we assume this is y_1 and z_2. We use c_1 on y_1 and z_2 and let $L'(v) = L(v) - c_1$ for each uncolored vertex v. Now we have $|Pot(L')| < |G \setminus \{y_1, z_2\}| = 6$. Since we have $|L'(z_1)| + |L'(y_2)| + |L'(z_3)| \geq 1 + 3 + 2 = 6$, we must have a common color c_2 (different from c_1) on two of z_1, y_2, and z_3. We use this color on these two vertices, then greedily color the remaining vertices of the net before coloring x_1 and x_2. (3) Observe that if $L(z_1)$ and $L(z_2)$ are disjoint, then (since $|Pot(L)| \leq 7$) either $L(z_1) \cap L(y_3) \neq \emptyset$ or $L(z_2) \cap L(y_3) \neq \emptyset$; in each case, we are in (2). Thus, if we are not in (1) or (2) above, then (again, since $|Pot(L)| \leq 7$) by symmetry we have $L(z_1) = \{a, b\}$, $L(z_2) = \{a, c\}$, $L(z_3) = \{b, c\}$, and $L(y_1) = L(y_2) = L(y_3) = \{d, e, f, g\}$. By symmetry, either $a \notin L(x_1)$ or $d \notin L(x_1)$. Thus, we use a on z_1 and z_2 and we use d on y_3. Now we greedily color z_3, y_1, y_2, x_2, x_1.

Let $G = K_2 * M$ and let x_1, x_2 denote the vertices of the K_2; for the M, let y_1 denote the 1-vertex, y_2 the 3-vertex, y_3 the 2-vertex adjacent to y_2, y_4 the 4-vertex, and y_5 and y_6 the remaining 2-vertices. By the Small Pot Lemma, $|Pot(L)| \leq 7$. Since $|L(y_1)| + |L(y_3)| + |L(y_6)| = 8$, two of them must have a common color c. If all three of y_1, y_3, y_6 have c, then we use c on all three, and afterward we color greedily y_2, y_4, y_5, x_1, x_2. So now we consider three cases. (1) If c appears in $L(y_3) \cap L(y_6)$, then we use c on y_3 and y_6, and let $L'(v) = L(v) - c$ for each uncolored vertex v. By the Small Pot Lemma, $|Pot(L')| \leq 5$. Since $|L'(y_1)| + |L'(y_4)| \geq 2 + 4 > 5$, we have a common color d (different from c) on y_1 and y_4. After we use d on y_1 and y_4, we color greedily y_2, y_5, x_1, x_2. (2) If c appears in $L(y_1) \cap L(y_3)$, then we use c on y_1 and y_3 and let $L'(v) = L(v) - c$ for each uncolored vertex v. Again we have $|Pot(L')| \leq 5$ and $|L'(y_2)| + |L'(y_5)| \geq 3 + 3 > 5$. After using a common color on y_2 and y_5, we greedily color y_4, y_6, x_1, x_2.

(3) Now suppose that c appears in $L(y_1) \cap L(y_6)$. If $c \in L(y_2)$, then we use c on y_2 and y_6, and let $L'(v) = L(v) - c$ for each uncolored vertex v. Again we have $|Pot(L')| \leq 5$ and $|L'(y_1)| + |L'(y_3)| + |L'(y_5)| \geq 1 + 3 + 2$ (since $c \notin L(y_3)$). So again we use a common color on two of y_1, y_3, and y_5, then greedily color the remaining vertices of the M before coloring x_1 and x_2. Suppose instead that $c \notin L(y_2)$. Now we use c on y_1 and y_6, and then use a common color on y_4 and y_5 (since $|Pot(L')| \leq 5 < 6 = 4 + 2 \leq |L'(y_2)| + |L'(y_5)|$). Finally, we greedily color y_3, y_4, x_1, x_2. □

Lemma 4.51. *The graph $K_2 * (B + K_t)$ is not d_1-choosable iff $K_2 * B$ is not d_1-choosable.*

Proof. Suppose $K_2 * B$ is not d_1-choosable and let L be a bad list assignment (not using the colors in $[t]$). To form a list assignment for $K_2 * (B + K_t)$, we start with L, then assign $[t]$ to each vertex in the K_t and add $[t]$ to the lists for the vertices in the K_2. Clearly $K_2 * (B + K_t)$ has no coloring from these lists.

Conversely, suppose $K_2 * B$ is d_1-chooable. Given a list assignment for $K_2 * (B + K_t)$, we greedily color the K_t; what remains is a list assignment for $K_2 * B$; thus, we can finish the coloring. □

Since $K_2 * 2P_3$ is d_1-choosable (Lemma 4.39) we see that any graph B such that $K_2 * B$ is not d_1-choosable must have at most one incomplete component.

Lemma 4.52. *If $K_2 * B$ is not d_1-choosable, then B consists of a disjoint union of complete subgraphs, together with at most one incomplete component H. If H has a dominating vertex v, then $K_2 * H = K_3 * (H - v)$, so by Lemma 4.34 we can completely describe H. Otherwise H is formed either by adding an edge between two disjoint cliques or by adding a single pendant edge incident to each of two distinct vertices of a clique. Furthermore, all graphs formed in this way are not d_1-choosable.*

Proof. Let B be a graph such that $K_2 * B$ is not d_1-choosable, and let H be the unique incomplete component of B. Suppose that H does not contain a dominating vertex. We first show that H is a tree of edge-disjoint cliques (clique tree), i.e., every cycle has an edge between every pair of its vertices. Since $K_2 * C_4$, $K_2 * C_5$, and $K_2 * P_5$ are d_1-choosable, we get that H has no induced C_4, C_5, or P_5; thus H is chordal. So if H is not a clique tree, then H contains an induced copy of K_4^-; call it D.

Let w denote a vertex adjacent to D. Each vertex adjacent to D can attach to the vertices of D in 8 possible ways (up to isomorphism); it can attach to 0, 1, or 2 of the vertices of degree 2, and also to 0, 1, or 2 of the vertices of degree 3 (but it must attach to at least one vertex), thus $3 * 3 - 1 = 8$ possibilites. Five of these possibilities yield a graph J such that $K_2 * J$ is d_1-choosable (since J contains an induced copy of either the antichair, $K_1 *$antipaw, $K_1 * P_4$, or C_4). So we consider the other three possibilities (these are the three possibilities when w is adjacent to both vertices of degree 3 in D).

If D is not dominating, then some vertex x is distance 2 from D, via w. In each case, the subgraph induced by D, w, and x contains an induced d_1-choosable subgraph (in two cases this is a antichair, and in the third case it is $K_1 *$ antipaw). Hence, D is dominating, and all of its neighbors are adjacent to both vertices of degree 3 in D. But now H has two dominating vertices. This contradicts our assumption that H has no dominating vertex. Hence, H is a clique tree.

Since H has no dominating vertex, it must contain an induced P_4, call it P. Since H has neither a P_5 nor a "chair" as an induced subgraph, each vertex adjacent to P must be adjacent to at least two vertices of P. Since C_4 and the antichair and $K_1 * P_4$ are all forbidden, each vertex adjacent to P is adjacent to exactly two consecutive vertices of P. Since both P_5 and the net are forbidden, every vertex in H is adjacent to P. Since $P_1 *$antipaw is forbidden, every pair of vertices that are adjacent to the same two vertices of P are also adjacent to each other. Finally, since M is forbidden, H must be formed in one of two ways. Either (a) begin with two disjoint cliques and add an edge between them, or else (b) begin with a clique and add exactly one edge incident to exactly two vertices of the clique. Furthermore, all graphs H formed by either (a) or (b) are such that $K_2 * H$ is not d_1-choosable.

In (a), suppose that we begin with a K_r and a K_s. We assign lists as follows: the K_r gets $[r]$, the K_s gets $\{r+1, \ldots, r+s\}$, the dominating vertices (on the other side of the join) get $[r+t]$; finally, the two endpoints of the additional edge also get α added to their lists. $K_2 * H$ is clearly not colorable from these lists, since all but one or $[r+t]$ must be used on H.

In (b), suppose that we begin with a K_r. We assign lists as follows: the K_r gets $[r]$, the two degree 1 vertices get $\{r+1, r+2\}$, the dominating vertices (on the other side of the join) get $[r+2]$; finally, the two vertices in the K_r that are endpoints of the pendant edges also get $r+1$ added to their lists. $K_2 * H$ is clearly not colorable from these lists, since all but one of $[r+2]$ must be used on H. □

4.4.4 Mixed list assignments

Lemma 4.53. *Let A be a graph with $|A| \geq 4$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and each component D of A has a vertex v such that $|L(v)| \geq d(v)$. Then L is good on G.*

Proof. By the Small Pot Lemma, $|Pot(L)| \le |A| + 1$. Say the E_2 has vertices $\{x, y\}$. Then $|L(x)| + |L(y)| \ge 2|A| - 2 > |A| + 1$ since $|A| \ge 4$. Coloring x and y the same leaves at worst a d_0 assignment L' on A where each component D has a vertex v with $|L'(v)| > d_D(v)$. Hence we can complete the coloring. □

Lemma 4.54. *Let A be a graph with $|A| \ge 3$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \ge d(v) - 1$ for all $v \in V(G)$, $|L(v)| \ge d(v)$ for some v in the E_2 and each component D of A has a vertex v such that $|L(v)| \ge d(v)$. Then L is good on G.*

Proof. By the Small Pot Lemma, $|Pot(L)| \le |A| + 1$. Say the E_2 has vertices $\{x, y\}$. Then $|L(x)| + |L(y)| \ge 2|A| - 1 > |A| + 1$ since $|A| \ge 3$. Coloring x and y the same leaves at worst a d_0 assignment L' on A where each component D has a vertex v with $|L'(v)| > d_D(v)$. Hence we can complete the coloring. □

4.5 Joins with K_1

Let G be a d_0-choosable graph. If $K_1 * G$ is not d_1-choosable, then we call G *bad*; otherwise we call G *good*. Adding a leaf to a graph does not change whether it is bad, so we focus on bad G such that $\delta(G) \ge 2$. We will also restrict our attention to connected bad graphs.

In this section, we apply Lemma 4.20 to characterize all bad triangle-free graphs. An easy special case of this classification for triangle-free graphs is the following lemma. We frequently use the idea of an independent set with a common color, so we call an independent set of size k with a common color an *independent k-set*.

Lemma 4.55. *If G is a connected bipartite graph with more edges than vertices, then $K_1 * G$ is d_1-choosable.*

Proof. Let A and B be the parts of G. Let L be a minimal bad d_1-assignment for $K_1 * G$. Since G has more edges than vertices, G has a cycle. Since G is also bipartite, G is d_0-choosable (by the classification of d_0-choosable graphs at the start of Section 4.2). By the Small Pot Lemma, $Pot(L) \le |G|$. Note that $\sum_{v \in A} d(v) = |E(G)| > |V(G)| \ge |Pot(L)|$. Similarly $\sum_{v \in B} d(v) > |Pot(L)|$. Now we apply Lemma 4.22 with $I_1 = A$ and $I_2 = B$. This proves the lemma. □

Lemma 4.56. *Let $\mathcal{C}$ be a collection of sets $I_1, \dots, I_k$, each of size 2. If for all $i \ne j$, we have $I_i \cap I_j \ne \emptyset$, then either there exists $v \in \cap_{i=1}^k I_i$ or there exist v_1, v_2, and v_3 such that each I_i equals either $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$.*

Proof. Suppose that $\cap_{i=1}^k I_i = \emptyset$. Consider distinct sets I_1 and I_2. Let $\{v_1\} = I_1 \cap I_2$, and let $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$. Since $\cap_{i=1}^k I_i = \emptyset$, there exists I_3 such that $v_1 \notin I_3$. So we must have $I_3 = \{v_2, v_3\}$. Now for all $k \ge 4$, we must have $|I_k \cap \{v_1, v_2, v_3\}| = 2$. □

The *core* of a graph is its maximum subgraph with minimum degree at least 2. Alternatively, it's the result if we repeatedly delete vertices of degree at most 1 for as long as possible.

Using Lemmas 4.20 and 4.56, we can prove the following classification.

Lemma 4.57. *If a graph G is bad, then $K_1 * G$ has a d_1-list assignment L such that one of the following 5 conditions holds.*

1. *L is a d-clique cover of G of size at most $|G|$.*
2. *There exists $v \in V(G)$ such that L is a d-clique cover of $G - v$ of size at most $|G| - 1$.*
3. *There exists a color c such that the union of all independent 2-sets in c induces P_4 and all other independent 2-sets are the end vertices of the P_4.*
4. *The union of all independent 2-sets is E_3 or E_2.*
5. *All independent 2-sets in L are the same color.*

Proof. Let z denote the K_1. We consider the possible ways for a bad list assignment L to satisfy Lemma 4.20. Clearly L has no independent k-sets, for $k \geq 3$. If L has no independent 2-sets, then Condition 1 holds. If all independent 2-sets in L are the same color, then Condition 5 holds. If L has only the same independent 2-set in multiple colors, then the 2-sets induce E_2, so Condition 4 holds. So instead L must have distinct independent 2-sets in distinct colors.

Assume that additionally all independent 2-sets intersect in a common vertex v. If $|Pot_{G-v}(L)| \leq |G| - 1$, then Condition 2 holds. So instead $|Pot_{G-v}(L)| \geq |G|$. So there exist some $w \in G - v$ and some color $c \in L(w)$ such that $c \notin L(z)$. By Lemma 4.20, G has an L-coloring that uses c on w and uses some other common color on two vertices of $G - w$. Now we can extend the coloring to z.

Now suppose that no vertex v lies in all independent 2-sets. If all independent 2-sets are distinct colors, then Lemma 4.56 implies that Condition 4 holds. Suppose we have two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_1, v_3\}$ in the same color c. Since L has no independent 3-set, v_2 is adjacent to v_3. Recall that L has an independent 2-set I_3 of another color c'. If $v_1 \notin I_3$, then I_3 is disjoint from either I_1 or I_2, so we can finish the coloring, by (2) in Lemma 4.20. Hence $v_1 \in I_3$. So the only independent 2-sets not containing v_1 must be of color c, say $\{v_2, v_4\}$. Since L has no independent 3-sets, we must have v_1 adjacent to v_4. Now we see that every independent 2-set in a color other than c must be $\{v_1, v_2\}$. This implies that v_2 and v_3 must be adjacent. Now Condition 3 holds.

Finally, suppose that L has two independent 2-sets $I_1 = \{v_1, v_2\}$ and $I_2 = \{v_3, v_4\}$ in a common color. If we are not in the case above, then $G[v_1, v_2, v_3, v_4] = C_4$. Now every independent 2-set I_3 of another color can intersect at most one of I_1 and I_2, so we can color the graph by (2) in Lemma 4.20. □

The classification in Lemma 4.57 is somewhat unsatisfying, since it does not immediately yield a method to construct all bad graphs of a certain size. In Lemma 4.58, we give a more satisfying characterizations for triangle-free bad graphs.

Lemma 4.58. *Let G be d_0-choosable and triangle-free. The graph $K_1 * G$ is not d_1-choosable iff the core of each component of G is an even cycle, except for at most one component which has a core that is either $\theta_{2,3,2l+1}$ (for some integer l) or is formed from a disjoint union of even paths by adding a vertex adjacent to all their endpoints.*

Proof. Suppose that H is d_0-choosable and triangle-free, but that $K_1 * H$ is not d_1-choosable. Let z denote the vertex of the K_1. Since H is d_0-choosable, no component of H can be a Gallai tree. Hence each component contains an even cycle. If H is a counterexample to the theorem, then some component D of H contains at least $|D|+1$ edges.

First suppose that there exist two components D_1 and D_2 of H with at least $|D_1|+1$ and $|D_2|+1$ edges, respectively. Let G_1 and G_2 be the cores of D_1 and D_2 and let L be a bad d_1-assignment for $K_1 * (G_1 + G_2)$. (We are guaranteed this bad list assignment from our bad d_1-list assignment for $K_1 * H$.) By greedily coloring G_2, we can get a bad d_1-assignment L_1 for $K_1 * G_1$. By the Small Pot Lemma, we may assume that $|Pot(L_1)| \le |G_1|$. Since $|E(G_1)| > |G_1|$ we get $\sum_{v \in G_1} |L(v)| = 2|E(G_1)| > 2|G_1| \ge 2|Pot(L_1)|$, so we have a color class α of size 3 in G_1, and hence an independent set of size 2 in G_1 with the common color α. By reversing the roles of G_1 and G_2, we can find an independent set of size 2 in G_2 with a common color β. Now we can apply (2) from Lemma 4.20. Thus, the core of all but at most one component D of H is an even cycle. Let G be the core of D and let L be a bad d_1-list assignment for $K_1 * G$.

We first prove that every color appears in the list of at most 3 vertices of G. Our plan is to either find an independent set of size 3 with a color common to its lists or to find two disjoint independent sets of size 2 with a distinct color common to the lists of each. Then we can apply (1) or (2) from Lemma 4.20.

Claim 1. *We may assume that* $|Pot_G(L)| = |G| - 1$.

Since G is d_0-choosable, we know that G has an L-coloring. If $|Pot_G(L)| < |L(z)| = |G| - 1$, then we can clearly extend the coloring to z. If there is at least one independent 2-set, then applying Lemma 4.21 gives $|Pot_G(L)| = |G|-1$. Otherwise, since G is triangle-free, $|G| = |E|$ and we are done.

Claim 2. *No color appears on 6 or more vertices of* G. Suppose the contrary. Since G is triangle-free (and since $R(3,3) = 6$), 3 of these vertices form an independent 3-set. Now we can apply (1) from Lemma 4.20.

Claim 3. *No color appears on 5 vertices of* G. Suppose that color α appears on exactly 5 vertices of G. Note that $\sum_{v \in V} |L(v)| = \sum_{v \in V} d_G(v) = 2|E(G)| \ge 2(|G|+1)$. By Claim 1, $|Pot(L)| = |G| - 1$. So by the Pigeonhole Principle (since $2(|G|+1|) = 2(|G|-1)+4$) there exists a color $\beta \ne \alpha$ such that β appears on at least 3 vertices in G. Since G is triangle-free, there exists an independent 2-set I with β as a common color. We may assume that the subgraph G_α induced by color α has no independent 3-set. Since G is triangle-free, G_α must be C_5. Now G_α has an independent 2-set that is disjoint from I. Thus we can apply (2) from Lemma 4.20.

Claim 4. *No distinct colors* α *and* β *each appear on 4 vertices of* G. Suppose that colors α and β each appear on 4 vertices of G and let G_α and G_β be the subgraphs induced by these colors. Since G_α is bipartite and has no independent 3-set, we can partition $V(G_\alpha)$ into two independent 2-sets I_1 and I_2. Similarly, we can partition $V(G_\beta)$ into independent 2-sets J_1 and J_2. Now we can finish by (2) from Lemma 4.20 unless each of I_1 and I_2 intersects each of J_1 and J_2. This implies that $V(G_\alpha) = V(G_\beta) = I_1 \cup I_2$. Thus, we can use α on I_1 and β on I_2.

Claim 5. *If* G *has a color class* α *of size 4 and a color class* β *of size 3, then* α *induces a* P_4*,* β *induces a* P_3*, and the* P_3 *and* P_4 *together induce a* C_5. Let G_α and G_β be the subgraphs induced by α and β. Let I be an independent 2-set in G_β. Since G_α is bipartite and has no independent 3-set, G_α is a subgraph of C_4 with at least two edges. If G_α is C_4, then let J_1 and J_2 denote the disjoint independent 2-sets in G_α. No independent set intersects both J_1 and J_2. Thus, we can apply (2) to I and some J_i. Similarly, if $G_\alpha = 2K_2$, then I is disjoint from some independent 2-set in G_α. Hence, $G_\alpha = P_4$ and I consists of the endpoints of the P_4. If G_β is not P_3, then we have a second choice for I, which cannot also be the endpoints of G_α. Thus, $G_\beta = P_3$. Since G is triangle-free, we get that $G_{\alpha\cup\beta} = C_5$.

Claim 6. *No color appears on 4 vertices of G.* By Claim 4, suppose that exactly one color, α, appears on 4 vertices of G. Let c_i denote the number of colors that appear on exactly i vertices. We have $2|E| = \sum_{v\in V} d(v) = \sum_{v\in V} |L(v)| = c_1+2c_2+3c_3+4(1)$. By Claim 1, we know that $|Pot(L)| = |G|-1$, so $c_1+c_2+c_3+1 = |G|-1$. Multiplying the second equation by 2 and subtracting it from the first gives $2(|E|-|G|) = c_3-c_1$. Let x and y denote the endpoints of G_α, as given by Claim 5. For each color class β of size 3, we can apply Claim 5. Thus each color in a class of size 3 appears on both x and y; so does color α. Hence $d(x) \geq 1 + c_3$ and $d(y) \geq 1+c_3$. Note that $2(|E|-|G|) = \sum_{v\in V}(d(v)-2) \geq (d(x)-2)+(d(y)-2) \geq 2c_3-2$; the first inequality holds because G is the core, and thus $\delta(G) \geq 2$. Combining this inequality with the equality above, we get $c_3 - c_1 \geq 2c_3 - 2$, which implies that $2 - c_1 \geq c_3$. Finally, this implies that $|E| - |G| \leq 1$.

If $|E| - |G| = 0$, then G is simply a 5-cycle, which is a Gallai tree, which yields a contradiction. Hence $|E| - |G| = 1$, which implies that $c_3 = 2$ and $c_1 = 0$. Thus $d(x) = d(y) = 3$. Since $|E| = |G| + 1$, $\delta(G) = 2$, and x and y lie on a common cycle, G must be a theta graph $\Theta_{2,3,k}$. If any color class of size 2 is an independent set I, then I must be $\{x, y\}$ (since otherwise we could use a common color on I and on two vertices of G_α); however, this is impossible, since we have already accounted for all of $L(x)$ and $L(y)$. Thus, every color class of size 2 must induce a K_2. Now a simple parity argument shows that the final path of the theta graph has odd length, so G is $\Theta_{2,3,2l+1}$.

Claim 7. *No vertex u is contained in every independent 2-set.* Suppose instead that such a u exists. By Claim 1, $|Pot(L)| = |G| - 1$. Since $\delta(G) \geq 2$, we get $\sum_{v\in(V-u)} d_G(v) \geq 2(|G| - 1)$. Since u appears in every independent 2-set (and G is triangle-free), each color appears at most twice on $G - u$. Since $|Pot_{G-u}(L)| \leq |Pot(L)| \leq |G| - 1$, every color in $|Pot_{G-u}(L)|$ appears exactly twice on $G-u$ and furthermore $d_{G-u}(v) = 2$ for all $v \in V(G)-u$. Hence, $G - u$ is a disjoint union of paths. Since each color class must induce a K_2 in $G - u$, we see that each path must be of odd length.

Claim 8. *No such G exists.* If there exist independent 2-sets I_1 and I_2, both with common color α, then by Claim 6, I_1 and I_2 intersect. Clearly, if independent 2-sets I_1 and I_2 have distinct common colors, then they must intersect (or we are done by (2) from Lemma 4.20). Thus, every pair of independent 2-sets intersects. Now by Claim 7 and Lemma 4.56, there exist vertices v_1, v_2, v_3 such that every independent 2-set is either $\{v_1, v_2\}$ or $\{v_1, v_3\}$ or $\{v_2, v_3\}$.

Now, similar to in Claim 6, we have $2|E| = c_1 + 2c_2 + 3c_3$ and $c_1 + c_2 + c_3 = |G| - 1$. Again $2(|E|-|G|) = c_3-c_1-2$. Now we have $2(|E|-|G|) = \sum_{v\in G}(d(v)-2) \geq (d(v_1)-2)+(d(v_2)-2)+(d(v_3)-2) \geq 2c_3-6$. So $c_3 - c_1 - 2 \geq 2c_3 - 6$, which implies that $c_3 \leq 4$, and hence $|E|-|G| \leq 1$. If $|E|-|G| = 0$, then G is a cycle, so we may assume that $|E|-|G| = 1$, which implies that $c_1 = 0$ and $c_3 = 4$.

Since $c_3 = 4$, there exist distinct colors α, β, and γ, each of which appear on independent

2-sets; say α appears on $\{v_1, v_2\}$, and β appears on $\{v_1, v_3\}$ and γ appears on $\{v_2, v_3\}$. Let w_1, w_2, and w_3 denote the third vertices with color α, β, and γ, respectively. If two (or all three) of the w_i coincide, then that vertex w has degree at least 3, so $\sum_{v \in V}(d(v) - 2) \geq 2c_3 - 5$. This implies that $c_3 \leq 3$, which contradicts our assumption that $c_3 = 4$. If instead w_1, w_2, and w_3 are distinct, then $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ induces a 6-cycle. Since the only two vertices in G of degree 3 lie on the 6-cycle, G is a theta graph. Suppose, without loss of generality, that v_1 and v_2 each appear in three color classes of size 3 and that v_3 appears in two of them. Now we have fully accounted for the colors in $L(v_1)$, $L(v_2)$, $L(v_3)$, and $L(w_1)$. However, we still need another color in each of $L(w_2)$ and $L(w_3)$. Since $c_1 = 0$, this gives a contradiction. We have now completed one direction of the proof. Below, we give the other direction.

Form G' from G by adding a pendant edge. Observe that $K_1 * G'$ is d_1-choosable iff $K_1 * G$ is d_1-choosable. Any d_1-list assignment will give a single color to the degree 1 vertex, so $K_1 * G'$ has a coloring from its lists iff $K_1 * G$ has a coloring from the resulting d_1-list assignment. Thus, given a graph H and its core G, the graph $K_1 * H$ is d_1-choosable iff $K_1 * G$ is d_1-choosable. To complete the proof, we need only provide list assignments to show that $K_1 * G$ is not d_1-choosable when G is a disjoint union of 4^+-cycles together with at most component that is $\Theta_{2,3,2l+1}$ or is formed from a disjoint union of odd paths by adding a vertex adjacent to all their endpoints.

For a k-cycle, we assign to each edge a distinct color and assign to each vertex the colors on its two incident edges. Since each color can be used only once in a proper coloring, every coloring of the k-cycle uses all k colors in its lists. Thus, if we let $L(z)$ contain $k - 1$ of those colors, then $K_1 * C_k$ has no L-coloring. Furthermore, for any graph G and any integer k, the graph $K_1 * (G + C_k)$ is d_1-choosable iff $K_1 * G$ is d_1-choosable.

Suppose that G is formed from disjoint even paths by adding a vertex v adjacent to all of their endpoints. We partition each path into copies of K_2 and give the vertices in each K_2 the same list, say $\{\alpha_i, \beta_i\}$. We use disjoint lists on each K_2 and we assign an arbitrary list of colors to vertex v. Finally, let $L(z) = \cup\{\alpha_i, \beta_i\}$. Any proper coloring of $G - u$ will use all the colors in $\cup\{\alpha_i, \beta_i\}$. Thus, z will have no color.

Finally, suppose that $G = \Theta_{2,3,2l+1}$. Let v_1, v_2, v_3, v_4, v_5 denote the vertices of the 5-cycle, where $d(v_1) = d(v_4) = 3$. Let $L(v_1) = L(v_4) = \{a, b, c\}$, $L(v_2) = L(v_3) = \{a, d\}$, and $L(v_5) = \{b, c\}$. Partition the $2l - 1$ path of $G \setminus \{v_1, \ldots, v_5\}$ into copies of K_2. As above, give the vertices in each copy of K_2 the same list $\{\alpha_i, \beta_i\}$; use disjoint lists on the K_2s. Now let $L(z) = \{a, b, c, d\} \cup (\cup\{\alpha_i, \beta_i\})$. Since each of $\{a, b, c, d\}$ must be used on $v_1, \ldots, v_5$, no color remains for z. □

Notation

Symbology	Meaning
$\lvert G\rvert$	the number of vertices G has
$\lVert G\rVert$	the number of edges G has
$G[S]$	the subgraph of G induced on S
$E_G(X, Y)$	the edges in G with one end in X and the other in Y
$E_G(X)$	$E_G(X, V(G) - X)$
$\chi(G)$	the chromatic number of G
$\omega(G)$	the clique number of G
$\alpha(G)$	the independence number of G
$\Delta(G)$	the maximum degree of G
$\delta(G)$	the minimum degree of G
$\kappa(G)$	the vertex connectivity of G
$\overline{G}$	the complement of G
$A + B$	the disjoint union of graphs A and B
$A * B$	the join of graphs A and B (that is, $\overline{\overline{A} + \overline{B}}$)
kG	$\underbrace{G + G + \cdots + G}_{k \text{ times}}$
G^k	$\underbrace{G * G * \cdots * G}_{k \text{ times}}$
$H \subseteq G$	H is a subgraph of G
$H \subset G$	H is a proper subgraph of G
$H \trianglelefteq G$	H is an induced subgraph of G
$H \triangleleft G$	H is a proper induced subgraph of G
$H \prec G$	H is a child of G
$f \colon S \hookrightarrow T$	an injective function from S to T
$f \colon S \twoheadrightarrow T$	a surjective function from S to T
$X := Y$	X is defined as Y
K_k	the complete graph on k vertices
E_k	the edgeless graph on k vertices (that is, $\overline{K_k}$)
P_k	the path on k vertices
C_k	the cycle on k vertices
$K_{a,b}$	the complete bipartite graph with parts of size a and b (that is, $E_a * E_b$)
$[n]$	$\{1, 2, \ldots, n\}$
$\mathbb{N}$	the natural numbers $(0, 1, 2, \ldots)$
$\mathbb{R}$	the real numbers

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